A GENERALIZED CONTOU-CARRÈRE SYMBOL AND ITS RECIPROCITY LAWS IN HIGHER DIMENSIONS

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Abstract. We generalize the theory of Contou-Carrère symbols to higher dimensions. To an $(n+1)$-tuple $f_0, \dots, f_n \in A((t_1))\cdots((t_n))^{\times}$, where A denotes a commutative algebra over a field k, we associate an element $(f_0, \dots, f_n) \in A^{\times}$, compatible with the higher tame symbol for $k = A$, and earlier constructions for $n = 1$, by Contou-Carrère, and $n = 2$ by Osipov–Zhu. Our definition is based on the notion of *higher commutators* for central extensions of groups by spectra, thereby extending the approach of Arbarello–de Concini–Kac and Anderson–Pablos Romo. Following Beilinson–Bloch–Esnault for the case $n = 1$, we allow A to be arbitrary, and do not restrict to artinian A. Previous work of the authors on Tate objects in exact categories, and the index map in algebraic K-theory is essential in anchoring our approach to its predecessors. We also revisit categorical formal completions, in the context of stable ∞-categories. Using these tools, we describe the higher Contou-Carrère symbol as a composition of boundary maps in algebraic K-theory, and conclude the article by proving a version of Parshin–Kato reciprocity for higher Contou-Carrère symbols.

1. Introduction

The Contou-Carrère symbol is a local invariant, in the sense of arithmetic, which arises from the geometry of the units of formal Laurent series $A((t))$. There are several competing perspectives on this symbol, and this paper aims to unify them, and extend this story to the *higher-dimensional* setting of $A((t_1))\cdots((t_n))$. Prior to the present article, the latter had only been achieved for $n \leq 2$.

To state our main results, let k be a field and A a k-algebra. Denote by ∂_i the boundary map in algebraic K-theory

$$\partial_i \colon K_{i+1}\left(A((t_1))\cdots((t_i))[[t_{i+1}]] \, , \, (t_{i+1}) \right) \to K_i\left(A((t_1))\cdots((t_{i-1}))[[t_i]] \, , \, (t_i) \right).$$

Let $\pi_* \colon K_1(A[[t_1]] \, , \, (t_1)) \to K_1(A)$ be the map induced by $A \to A[[t_1]]$, and let $\det \colon K_1(A) \to A^{\times}$ be the determinant. The following theorem is a primary application of our earlier results [BGW14a].

Theorem 1.1. *The units $A((t_1))\cdots((t_n))^{\times}$ have a canonical central extension by the spectrum $\Omega\,\mathbb{K}_A$ (see Definition 3.9). For $f_0, \dots, f_n \in A((t_1))\cdots((t_n))^{\times}$ we define the* Contou-Carrère symbol *to be the corresponding higher commutator $(f_0, \dots, f_n)$.*

(a) *For $n \leq 2$ this recovers the definitions of Contou-Carrère and Osipov–Zhu.*
(b) *Further, we have*

$$(f_0, \dots, f_n)^{(-1)^n} = \det \circ \pi_* \circ \partial_1 \circ \cdots \circ \partial_n \{f_0, \dots, f_n\},$$

relating the commutator with Loday's cup products/higher Steinberg symbols in algebraic K-theory.

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To give this statement meaning, one needs to clarify what is meant by a *central extension* of a group by a *spectrum*, and define higher commutators in this context, generalising the classical treatment of central extensions and commutators described below. We introduce such

$$\text{spectrum} \longrightarrow \widehat{G} \longrightarrow G \qquad\qquad \text{(spectral “central extensions”)}$$

in this paper. Assertion (a) then amounts to the verification that this new theory of spectral extensions and higher commutators extends the classically known cases. In order to develop this, we work in the context stable homotopy theory.

Theorem 1.2. *Examples of higher commutators $f_0 \star \cdots \star f_n$ of spectral extensions include:*

(i) *Loday's Steinberg symbols $\{f_0, \dots, f_n\} \in K_{n+1}(R)$, for R a commutative ring,*
(ii) *Contou–Carrère's original symbol (f_0, f_1) for $f_0, f_1 \in A((t))^\times$,*
(iii) *and Osipov–Zhu's Contou-Carrère symbol (f_0, f_1, f_2) for $f_0, f_1, f_2 \in A((s))((t))^\times$, as defined in* [OZ13].

See Propositions 3.31 and 3.3.4 for proofs of these statements. The last two facts motivated our definition of the general higher Contou-Carrère symbols.

For concreteness, we now state two special cases of Theorem 1.1. The statement of the first requires no knowledge of higher commutators, and, to our knowledge, is a new result in the literature, though it had been conjectured by Kapranov–Vasserot in [KV07].

Theorem 1.3. *Let k be a field, and let A be a k-algebra. The classical Contou-Carrère symbol factors through the boundary map in K-theory*

$$\begin{tikzcd} A((t))^\times \times A((t))^\times \arrow[r, "(-,-)^{-1}"] \arrow[d, "\{-,-\}"'] & A^\times \\ K_2(A((t))) \arrow[r, "\partial"] & K_1(A) \arrow[u, "\det(-)"'] \end{tikzcd}$$

or, in equations, $(f,g)^{-1} = \det(\partial(\{f,g\}))$. Here $(-,-)$ and $-,-$ refer to the classical commutator and classical Steinberg symbol respectively.

A second proof of this case has recently appeared in Osipov–Zhu [OZ13].

Passing to dimension 2, we encounter Osipov–Zhu's two-dimensional symbol. This also fits into our framework, and the precise statement for Theorem 1.2 in this case reads:

Theorem 1.4. *Let $\mathbf{P}$ be a Picard groupoid and let $\phi\colon G \longrightarrow B\mathbf{P}$ be the monoidal map corresponding to a central extension of G by $\mathbf{P}$. We denote by $e\colon \Sigma^\infty BG \longrightarrow \Sigma\mathbb{B}\mathbf{P}$ the corresponding spectral extension of G by $\Omega\mathbb{B}\mathbf{P}$, the spectrum associated to the Picard groupoid $\mathbf{P}$. Then,*

$$C_3(f,g,h) = f \star g \star h,$$

where C_3 refers to the generalized three-variable commutator of Osipov-Zhu.

See Section 3.4.2 for the proof, which rests on relating our stable homotopy perspective with the categorical model employed by Osipov–Zhu.

Our next result is a type of adelic “reciprocity law”: Let X be an n-dimensional k-variety and $0 \le i \le n$.

Theorem 1.5. *For $f_0, \dots, f_n \in A^\times_{X,\zeta}$ the product of Contou-Carrère symbols, over all $Z_{i-1} \subset Z \subset Z_{i+1}$ is well defined, and we have*

$$\prod_Z (f_0, \dots, f_n)_{\zeta_Z} = 1.$$

Here $\zeta = \{Z_j\}_{j=1,j\neq i}^n$, called an *almost saturated flag*, is a collection of j-dimensional closed subschemes $Z_j \subset X$ such that $Z_j \subset Z_k$ for $j < k$. $A_{X,\zeta}$ denotes a ring formed as an iterated completion of $A(X)$ (the A-valued rational functions on X) at the places $Z_j \times_k \operatorname{Spec}(A)$ (cf. Definition 6.2). As for classical adèles, the ring $A_{X,\zeta}$ is built from rings A_{X,ζ_Z}, one for each i-dimensional closed subset

$$Z_{i-1} \subset Z \subset Z_{i+1}.$$

Each of these rings carries a higher Contou-Carrère symbol $(f_0, \dots, f_n)_{\xi_Z}$, and the geometry of $A_{X,\zeta}$ gives rise to the above relationship between these symbols.

This theorem extends results for X of dimension one by Anderson–Pablos Romo [APR04] and Pál [Pál10] (for A 0-dimensional), Beilinson–Bloch–Esnault [BBE02] (for A arbitrary), and results for X of dimension 2 by Osipov–Zhu [OZ13]. We can paraphrase the statement of Theorem 1.1 (a) as saying that we extend the theory of the Contou-Carrère symbol from dimension ≤ 2 to the general case; statement (b) gives a K-theoretic description of the symbol, and Theorem 1.5 establishes the reciprocity law for these symbols.

The classical perspective. To put this work into context, we review the history of the Contou-Carrère symbol. In [CC94], Contou-Carrère defined his eponymous symbol as a bilinear pairing

$$(-,-)\colon L\,\mathbb{G}_m \times L\,\mathbb{G}_m \longrightarrow \mathbb{G}_m,$$

where $\mathbb{G}_m$ denotes the multiplicative group, $\mathbb{G}_m(A) = A^\times$, and $L\,\mathbb{G}_m(A) = A((t))^\times$ is by definition the loop group of $\mathbb{G}_m$. Contou-Carrère proved that the symbol induces an equivalence between $L\,\mathbb{G}_m$ and its Cartier dual $L\,\mathbb{G}_m^* = \mathsf{Hom}_{\mathrm{grp}}(L\,\mathbb{G}_m, \mathbb{G}_m)$, and thereby established a local analogue of the self-duality (in the sense of abelian varieties) of the Jacobian of a complete smooth curve. Using a presentation

$$f = \prod_{-\infty}^{i=-1}(1 - a_i t^i) a_0 t^{\nu(f)} \prod_{i=1}^{\infty}(1 - a_i t^i),$$

and

$$g = \prod_{-\infty}^{i=-1}(1 - b_i t^i) b_0 t^{\nu(g)} \prod_{i=1}^{\infty}(1 - b_i t^i),$$

one obtains the expression

$$(f,g) = (-1)^{\nu(f)\nu(g)} \frac{a_0^{\nu(g)} \prod_{i=1}^\infty \prod_{j=1}^\infty (1 - a_i^{j/(i,j)} b_{-j}^{i/(i,j)})^{(i,j)}}{b_0^{\nu(f)} \prod_{i=1}^\infty \prod_{j=1}^\infty (1 - a_{-i}^{j/(i,j)} b_j^{i/(i,j)})^{(i,j)}}. \tag{1}$$

In the special case that $A = k$, this is equivalent to the tame symbol

$$(f,g) = (-1)^{\nu(f)\nu(g)} \frac{f^{\nu(g)}}{g^{\nu(f)}}|_{t=0},$$

for $f, g \in k((t))^\times$, and the general Contou-Carrère symbol can be understood as a deformation of the tame symbol over the base $\operatorname{Spec}(A)$.

Viewpoints on symbols. The type of invariant going by the name of a "symbol" appears in number theoretic, functional analytic and algebro-geometric contexts. Here is an excerpt of this dictionary:

Archimedean counterparts	Non-archimedean counterparts
polarized Hilbert spaces $\mathcal{H}$	Tate vector spaces $k((t))$
norm topology	linear topology
Fredholm operators $\mathrm{GL}_{res}(\mathcal{H},\mathcal{H}^+)$	automorphisms $Aut_{Tate}k((t))$
Segal-Wilson Grassmannian	Sato Grassmannian
Toeplitz operators	multiplication operators
Fredholm index	index (in the sense of [BGW2])
Helton-Howe/Brown symbol	tame symbol
$f,g\mapsto \exp\left(\frac{1}{2\pi i}\int\int \frac{\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}}{fg}\mathrm{d}\mu\right)$	$f,g\mapsto(-1)^{v(f)v(g)}\left(\frac{f^{v(g)}}{g^{v(f)}}\right)(0)$
?	Contou-Carrère symbol

We refer the reader to [BGW14a] for more background on these analogies. Recent work of Horozov and Luo [HL13] suggests that the "archimedean" counterpart of the Contou-Carrère symbol (also the generalization we introduce in this paper) could be a type of iterated integral in the sense of Chen. Historically, the first kind of symbols of course came from number theory, but let us stick to the above picture. In the literature ideas have been wandering between these viewpoints, inspiring the other, and sometimes ideas were rediscovered independently. Let us look at an example:

In functional analysis, a theorem of F. Noether states that Toeplitz operators, a cut-off from multiplication operators, can compute the winding number of the function (and vice versa). Later, Tate introduced a philosophically equivalent concept as the basis of his treatment of residues on curves in algebraic geometry [Tat68]. We alert the reader that since Tate wanted a theory working over all base fields, one needs to have an algebraic replacement for Hilbert space. These are the Tate vector spaces already mentioned above in the dictionary. The essence of Tate's approach amounts to viewing the multiplication operator

$$k((t))\xrightarrow{T_f}k((t))$$
$$g\mapsto fg$$

for $f\in k((t))^\times$ as an analogue of a Toeplitz operator. Like classical Toeplitz operators, these operators determine Fredholm operators on the sub-space $k[[t]]$, and, again like classical Toeplitz operators, if T_f and T_g commute, then one can define a symbol

$$\{f,g\}\in k^\times$$

measuring the "analytic joint torsion" of the associated operators on $k[[t]]$.[1] Tate actually treated an additive version of this story, but inspired by his work, Arbarello–de Concini–Kac [ADCK89] identified $\{f,g\}$ with Weil's tame symbol.

To extend the additive version of the story to dimension n, Beilinson in [Bei87] and [Bei80] replaced the classical Tate vector spaces (or conventional polarized Hilbert spaces) by "n-Tate spaces". This extension also provides the context for the present work. We rely heavily on our foundational work in [BGW14b], which develops the necessary framework on the right-hand side of the above table in higher dimensions, and [BGW14a], where we develop the index theory of the relevant Toeplitz-style operators. In the present work, we apply this to define and study higher symbols, compatible to the viewpoint as analytic joint torsion on the left-hand side of the above table. An analogue of n-Tate spaces on the Hilbert space side is presently not known for $n>1$.

[1]In functional analysis this was pioneered by Helton–Howe [HH75] and even appears in a textbook treatment by Rosenberg [Ros94, 4.4.24]; more recently it has been studied by Carey–Pincus [CP99] and Kaad–Nest [KN].

Symbols via central extensions. The idea that central extensions should relate to symbols can be traced back to Tate's paper [Tat68] on *residues and differentials on algebraic curves.* Inspired by Tate's ideas, Arbarello–de Concini–Kac proved in [ADCK89] that up to sign, the tame symbol can be described as a commutator pairing for a canonical central extension of the group $k((t))^\times$. In order to define higher symbols, we introduce central extensions of groups by spectra and develop a calculus of higher commutators associated to these central extensions.

Recall that a central extension of a group G by an abelian group A

$$e\colon 1 \longrightarrow A \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1,$$

allows one to define a commutator pairing $(f, g)_e \in A$, for any two elements $f, g \in G$ with $[f, g] = 1$, by choosing lifts $\widetilde{f}, \widetilde{g}$ of f and g and defining

$$(f, g)_e := [\widetilde{f}, \widetilde{g}] \in A.$$

In [APR04], Anderson and Pablos Romo showed that the Contou-Carrère symbol can also be obtained as a commutator pairing (up to sign), as long as we assume the k-algebra A to be Artinian. This restriction was eventually removed by Beilinson–Bloch–Esnault [BBE02], who studied symbols in analogy with ϵ-factors.

To study 2-dimensional symbols, Osipov and Zhu [OZ11] introduced central extensions of groups by Picard groupoids. They developed a calculus of higher commutators associated to such extensions, and identified the 2-dimensional tame symbol as the commutator of a central extension of the group $k((t_1))((t_2))^\times$ by the Picard groupoid of graded k-lines. In [OZ13], they constructed an analogous extension of the group $A((t_1))((t_2))^\times$, and they defined the 2-dimensional Contou-Carrère symbol as the commutator of this extension.

Our approach begins with the observation that a Picard groupoid can be viewed as a particulary simple form of spectrum (in the sense of stable homotopy theory). Following this, we introduce in Section 3 the notion of a central extension of a group by a spectrum and we develop a calculus of higher commutators for spectral extensions which extends Osipov and Zhu's. This calculus for spectral extensions provides the technical framework for defining and computing higher dimensional symbols, and it forms the starting point for Theorem 1.1.

To summarise the preceding paragraphs, the 1-dimensional Contou-Carrére symbol is obtained as a commutator pairing for the Kac-Moody graded central extension of the loop group $L\,\mathbb{G}_m$, while the 2-dimensional Contou-Carrère symbol is obtained as a commutator pairing for the "Kac-Moody" central extension of the double loop group $L^2\,\mathbb{G}_m$ by the Picard groupoid of graded lines. Starting from the observation that the Picard groupoid of graded lines is the 1-truncation of algebraic K-theory, we construct "Kac-Moody" central extensions of the n-fold loop group $L^n\,\mathbb{G}_m$ by the looping $\Omega\,\mathbb{K}$ of the K-theory spectrum, and we define the n-dimensional Contou-Carrère symbol as the higher commutator of this extension.

Symbols and K-theory. Theorem 1.1 has already indicated how closely related the Contou-Carrère symbol is to boundary maps in algebraic K-theory. The analogous relationship for tame symbols has long been known. The first such result of which we are aware is in Milnor's [Mil70], where he shows that for a discrete valuation ring R with fraction field F, and residue field κ, one obtains the tame symbol for $f, g \in F^\times$, by applying the boundary map $\partial\colon K_2(F) \longrightarrow K_1(\kappa)$ to the Steinberg symbol $f \cup g \in K_2(F)$.

The description as boundary map forms the basis of Kato's approach in [Kat86], where he defines the higher tame symbol of $f_0, \ldots, f_n \in k((t_1))\cdots((t_n))^\times$ by

$$(f_0, \ldots, f_n)^{(-1)^n} := \partial_1 \circ \cdots \circ \partial_n\{f_0, \cdots, f_n\}),$$

where $\partial_i\colon K_{i+1}(k((t_1))\cdots((t_i))) \longrightarrow K_i(k((t_1))\cdots((t_{i-1})))$ is again the corresponding boundary map in algebraic K-theory.

As remarked earlier, the K-theoretic viewpoint on tame symbols has to date lacked a counterpart for Contou-Carrère symbols. However, taking the functional analytic interpretation of this symbol alongside the well-known relationship between index theory of Fredholm operators and topological

K-theory, one should expect algebraic K-theory to play a role. Part of our motivation for this work was to provide this, not least because it essentially reduces the reciprocity law to the fact that $d^2 = 0$ in a Gersten-style complex (see Section 6 below). We also hope that this will have consequences beyond the theory of Contou-Carrère symbols. As Beilinson, Bloch and Esnault observe in [BBE02], the Contou-Carrère symbol can be viewed as an invariant of zeroth order differential operators. If one defines the analogous invariant for first order operators, one obtains (de Rham) ϵ-factors, and the proof of reciprocity carries over to give the product rule. Having developed the 1-dimensional story, one is naturally led to ask

Problem 1.6. [BBE02, 1.7.c] *Produce a geometric theory of de Rham ϵ-factors for n-dimensional local fields, and establish a product formula in analogy with the one dimensional case.*

It is reasonable to hope that the higher dimensional Contou-Carrère symbol and its reciprocity law may allow for progress on this problem.

Derived completion. The finite dimensionality of the cohomology of a proper curve provides a key geometric input in proving the reciprocity law for 1-dimensional symbols. In the setting of higher dimensional reciprocity laws, we can morally interpret the ring $A_{X,\zeta}$ of Theorem 1.5 as the ring of A-valued rational functions of an exotic "curve" X_ζ associated to the almost saturated flag $\{Z_j\}_{j \neq i}$ of subset of X. In principle, this "curve" should be obtained by iteratively completing X at the Z_j and then removing the special point Z_j. However, the theories of Berkovich or rigid analytic spaces are insufficient to handle such constructions. Rather than develop such a theory, we take a non-commutative geometry approach and replace X by its stable ∞-category of perfect complexes. The operations of localization and completion of schemes have analogues for stable ∞-categories, cf. Thomason–Trobaugh [TT90] (localization) and Efimov [Efi10] (completion). We apply these in Section 5 to construct a stable ∞-category which plays the role of "$\mathrm{Perf}(X_\zeta)$" and we use the (non-commutative) "geometry" of this stable ∞-category to deduce the reciprocity law.

Quick summary of the proof of reciprocity. The reciprocity law of Theorem 1.5 expresses information about the local geometry of a variety around an almost saturated flag. As remarked above, our approach to higher symbols allows us to reduce the reciprocity law to the statement that $d^2 = 0$ in a Gersten-style complex. As with the classical Gersten complex, the differentials arise as (sums of) boundary maps in K-theory localization sequences. Our work on derived completion allows us to obtain these localization sequences in our setting and deduce reciprocity.

We now explain the strategy of this proof in the case $A = k$ and $n = 2$. Let Y be a smooth surface over k and $x \in Y$ a closed point. For a triple of non-zero elements f, g, h of the fraction field of the completed ring $\widehat{\mathcal{O}}_{Y,y}$ we must show that the product

$$\prod_C \{f, g, h\}_{C,x}$$

ranging over curves containing x, is well-defined and equals 1. Without loss of generality we may replace Y by $Y' = \operatorname{Spec} \widehat{\mathcal{O}}_{Y,x}$, and apply the following argument. There exists a closed subset $Z \subset Y'$, such that Z is a union of curves containing x, and f, g, h are regular elements on $U = Y' \setminus Z$. Our results above identify this product with a composition of boundary maps as in the lower path of the diagram

$$\begin{tikzcd} K_3(U) \arrow[r, "\partial"] & K_2(Y' \setminus \{x\}, Z \setminus \{x\}) \arrow[r] \arrow[dr, "\partial"'] & K_2(Y' \setminus \{x\}) \arrow[d, "\partial"] \\ & & K_1(\{x\}). \end{tikzcd}$$

However, this is also equivalent to the upper path of the diagram, the last two maps of which are successive maps in a long exact sequence.

For dimension $n > 2$, we employ an analogous argument. However, we must now replace the punctured formal surface $\operatorname{Spec} \widehat{\mathcal{O}}_{Y,x} - \{x\}$ with a more exotic object obtained by completing and removing at all the closed subsets in an almost saturated flag. Our treatment of derived completions supplies us with the necessary localization sequences in this setting, while our treatment of symbols allows us to identify the appropriate product with a composition of boundary maps from these sequences. It is then a relatively straightforward matter to show that this composition is zero when restricted to tuples of invertible elements of $A_{X,\zeta}$.

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2. Preliminaries

2.1. **∞-Categories.** We briefly review the main ideas from the theory of ∞-categories that will be repeatedly used in our work. For a more detailed overview, we refer the reader to Groth's survey [Gro10].

2.1.1. *Spaces are ∞-Groupoids.* The only topological spaces that play a role for us are those which are homotopy equivalent to a CW-complex. The term *space* will always refer to those topological spaces. Since every space X is weakly equivalent to the geometric realization of the simplicial set of singular simplices $S_\bullet(X)$, we could equivalently work with simplicial sets.

We now remind the reader of a hierarchy on the category of spaces.

- A *homotopy* 0*-type* is a space homotopy equivalent to a discrete space,
- a *homotopy* 1*-type* is a space with vanishing higher homotopy groups,
- a *homotopy n-type* is a space X with $\pi_k(X) = 0$ for $k \geq n+1$.

The category of homotopy 0-types is equivalent to the category of *sets*. The category of homotopy 1-types seems closely related to the category of (small) groupoids $\mathcal{G}$. To a groupoid $\mathcal{G}$, one simply assigns the geometric realization of its nerve $|N\,\mathcal{G}\,|$. Vice versa, given a space X, we have the Poincaré groupoid $\pi_{\leq 1}(X)$. Its set of objects is the set of points in X. A morphism from $x \in X$ to $y \in X$ is a homotopy class of paths connecting x and y.

The natural map of groupoids $\mathcal{G} \longrightarrow \pi_{\leq 1}(|N\,\mathcal{G}\,|)$ is not a *strict isomorphism*. However, it is an *equivalence* of groupoids. Using this fact, one can show that the above functors *induce an equivalence between the* 2*-category of* groupoids *and the* 2*-category of* homotopy 1-types. This motivates the following slogan of modern homotopy category:

> *The collection of homotopy n-types forms the $(n+1)$-category of n-groupoids. Spaces correspond to ∞-groupoids.*

2.1.2. *Simplicial Sets and ∞-Categories.* Intuitively speaking, an ∞-category C is a category enriched in ∞-groupoids (i.e. spaces). Hence, for every pair of objects $X, Y \in \mathsf{C}$ we have a space of morphisms $\mathsf{Hom}_{\mathsf{C}}(X,Y)$. Since this space will only matter up to homotopy, composition should not be expected to be defined strictly, but only up to a homotopy, which itself is well-defined up to higher homotopies of all orders. It is difficult to extract a meaningful definition from this heuristic description, but its value should not be underestimated. To a large extent it is possible to work with ∞-categories as a blackbox, as long as one accepts that there is a well-behaved calculus of homotopy coherent commutative diagrams.

In the rigorous setting of quasi-categories (see e.g. Lurie's [Lurb]), one defines ∞-categories as simplicial sets satisfying a mild technical condition. This definition is motivated by the classical construction of nerves of categories. Recall that for a classical category C we define its nerve $N\mathsf{C}$ to be the simplicial set with objects as 0-simplices, morphisms as 1-simplices, composable pairs of morphisms as 2-simplices, etc. Grothendieck observed that one can reconstruct a category from its nerve (even up to isomorphism of categories, see e.g. [Lurb]). A simplicial set is the nerve of a

category, if and only if it satisfies a collection of strict horn-filling conditions, the most important one of which is explained below.

The set of 2-simplices of $N\mathsf{C}$ can be understood as the set of commuting triangles

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "gf"] & & Z. \end{tikzcd}$$

The horn-filling condition in this particular case amounts to stating that every diagram

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] & & Z. \end{tikzcd}$$

can be completed to a commuting triangle as above. For a classical category this can always be achieved in precisely one way.

Even if one does not know the definition of an ∞-category, one could try to guess what the nerve of an ∞-category should be. Accepting the above slogan that, whatever ∞-categories are, we want to have a good calculus of commutative diagrams, we arrive at the set of commuting triangles

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z. \end{tikzcd}$$

as a definition for the set of 2-simplices in the nerve. There are two interesting new features. First of all we cannot say that h is the composition of f and g. Rather, h is one of possibly many compositions of f and g. The invisible 2-cell of the triangle above should be thought of as a homotopy connecting both sides. It turns out that if we no longer require horns to be filled uniquely, this is sufficient to characterize nerves of ∞-categories. This is precisely how quasicategories are defined by Joyal and in [Lurb].

What separates the subcategory of classical categories from its complement in quasicategories is the existence of a *strict* composition operation for morphisms. In ∞-categories, composition is only well-defined up to a contractible space of choices. It is this little bit of extra homotopical glue, which makes the theory of ∞-categories so flexible.

As a natural consequence of this liberality, the only possible notion of *commutative* diagrams is automatically *homotopy coherent* in a strong sense.

If $I_\bullet$ is a simplicial set, then an $I_\bullet$-indexed commutative diagram in an ∞-category C is a map of simplicial sets $I_\bullet \longrightarrow \mathsf{C}$. A commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

for example is a map of simplicial sets $(\Delta^1)^2 \longrightarrow \mathsf{C}$, sending the 0-simplices of the square $(\Delta^1)^2$ to the objects X, Y, Z, W.

2.2. **Index Theory in Algebraic K-Theory.**

2.2.1. *Tate Objects in Exact Categories.* We recall the constructions of Ind, Pro, and Tate objects in exact categories, and refer the reader to [BGW14b] for more details and proofs. The ideas of these constructions go back to papers by Beilinson [Bei87] and Kato [Kat00], and have also been studied by Previdi in [Pre11]. We also refer the reader to Drinfeld's theory of Tate R-modules introduced in [Dri06].

A *filtered set* I is a set I together with a partial ordering $\leq$, such that for each pair $(i,j) \in I^2$ there exists a $k \in I$, satisfying $i \leq k$ and $j \leq k$. Every filtered set can be viewed as a category in a straightforward manner.

Let C be an exact category. An *admissible* Ind-object in C indexed by I is a functor $X\colon I \longrightarrow \mathsf{C}$, such that the relation $i \leq j$ determines an admissible monomorphism with respect to the exact structure of C.

For example, we can take I to be the set $\mathbb{N}$ with its natural ordering. An $\mathbb{N}$-indexed admissible Ind-object in C can then be pictured as a formal colimit of a diagram

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots . \tag{2}$$

Every admissible Ind-object gives rise to a left exact presheaf. To $X\colon I \longrightarrow \mathsf{C}$ one associates the presheaf

$$A \mapsto \varinjlim_{i \in I} \mathsf{Hom}(A, X(i)).$$

The resulting full subcategory of $\mathsf{Lex}(\mathsf{C})$ of all objects of this shape is denoted by $\mathsf{Ind}^a(\mathsf{C})$. In Theorem 3.7 of [BGW14b] the authors showed that $\mathsf{Ind}^a(\mathsf{C})$ is an extension closed sub-category of $\mathsf{Lex}(\mathsf{C})$. This implies that it inherits a structure of an exact category.

Admissible Pro-objects in C are defined dually, i.e. by replacing the role of admissible monomorphisms by admissible epimorphisms. In short we have, $\mathsf{Pro}^a(\mathsf{C}) = (\mathsf{Ind}^a(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}$. An admissible Pro-object indexed by a filtered set I is a functor $X\colon I^{\mathrm{op}} \longrightarrow \mathsf{C}$, which sends $i \leq j$ to an admissible epimorphism in C. For $I = \mathbb{N}$ we obtain the dual depiction of a Pro-object as a formal limit of a diagram

$$X_0 \twoheadleftarrow X_1 \twoheadleftarrow X_2 \twoheadleftarrow \cdots . \tag{3}$$

An *elementary Tate object* is an admissible Ind-Pro-object, i.e. an object V in $\mathsf{Ind}^a\mathsf{Pro}^a(\mathsf{C})$, which can be (non-canonically) written as an extension

$$L \hookrightarrow V \twoheadrightarrow V/L, \tag{4}$$

with $L \in \mathsf{Pro}^a(\mathsf{C})$ and $V/L \in \mathsf{Ind}^a(\mathsf{C})$. We refer to any such L as a *lattice* in V. The category of elementary Tate objects in C has a natural exact structure (Theorem 5.4 in [BGW14b]), and will be denoted by $\mathsf{Tate}^{el}(\mathsf{C})$.

Every diagram $\mathbb{N} \times \mathbb{N} \longrightarrow \mathsf{C}$ of the form

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
X_{10} \arrow[u, hook] & X_{11} \arrow[l, two heads] \arrow[u, hook] & X_{12} \arrow[l, two heads] \arrow[u, hook] & \cdots \arrow[l, two heads] \\
X_{00} \arrow[u, hook] & X_{01} \arrow[l, two heads] \arrow[u, hook] & X_{02} \arrow[l, two heads] \arrow[u, hook] & \cdots , \arrow[l, two heads]
\end{tikzcd} \tag{5}$$

such that all squares are bicartesian, induces an object in $\mathsf{Tate}^{el}(\mathsf{C})$. Comparing this diagram with (2) and (3), we see that every row yields a presentation of an admissible Pro-object in C, and every column an admissible Ind-object. In fact, one sees that every row in the diagram (5) induces a lattice in the resulting Tate object.

The exact category $\mathsf{Tate}(\mathsf{C})$ of *Tate objects* in C is defined to be the idempotent completion of $\mathsf{Tate}^{el}(\mathsf{C})$. If R is a ring, and $\mathsf{C} = P_f(R)$, the exact category of finitely-generated projective R-modules, then $\mathsf{Tate}(P_f(R))$ contains Drinfeld's category of Tate R-modules as a full subcategory. See [BGW14b, Thm. 5.26], where we show that for countable index sets I, the two categories are in fact equivalent. We emphasize that in [Dri06], Drinfeld refers to what we call lattices as *co-projective lattices*.

Definition 2.1. *For a category D (respectively ∞-category), we denote by $\mathsf{D}^\times$ the maximal groupoid contained in D (respectively ∞-groupoid).*

The following result is [BGW14a, Prop. 3.3]. upper bound ([BGW14b, Thm. 6.7]).

Proposition 2.2. *For an idempotent complete exact category C, we denote by $Gr^{\leq}_\bullet(\mathsf{C})$ the simplicial object in groupoids, which parametrizes chains $(V \supset L_n \supset \cdots \supset L_0)$, where V is an elementary Tate object in C, and each L_i is a lattice in V. We have a forgetful morphism $Gr^{\leq}_\bullet(\mathsf{C}) \longrightarrow \mathsf{Tate}^{el}(\mathsf{C})^\times$, which induces an equivalence $|Gr^{\leq}_\bullet(\mathsf{C})| \xrightarrow{\cong} \mathsf{Tate}^{el}(\mathsf{C})^\times$.*

We refer the reader to *loc. cit.* for a detailed proof. It is a formal consequence of the main result of [BGW14b] which states that the partially ordered set of lattices $Gr(V)$ is filtered, i.e. every finite collection of lattices has a common

2.2.2. *The Index Map.* Let C be an exact category, following Waldhausen [Wal85] we denote by $S_n(\mathsf{C})$ the exact category, whose objects correspond to chains

$$X_1 \hookrightarrow \cdots \hookrightarrow X_n$$

of admissible monomorphisms. We refer the reader to *loc. cit.* for the definition of face and degeneracy maps, which allow one to define a simplicial object $S_\bullet(\mathsf{C})$ in the 2-category of exact categories.

Waldhausen's treatment of algebraic K-theory in [Wal85] implies that, for an exact category C, the classifying space BK_{C} is equivalent to the geometric realization of the simplicial object in groupoids $|S_\bullet \mathsf{C}^\times|$.

Now let C be an idempotent complete exact category, and let $Gr^{\leq}_\bullet(\mathsf{C})$ be as in Proposition 2.2.

Definition 2.3. *Let $\mathsf{Index}\colon Gr^{\leq}_\bullet(\mathsf{C}) \longrightarrow S_\bullet(\mathsf{C})$ be the map sending $(V \supset L_n \supset \cdots \supset L_0)$ to $(L_1/L_0 \hookrightarrow \cdots \hookrightarrow L_n/L_0)$. The geometric realization $\mathsf{Tate}^{el}(\mathsf{C})^\times \longrightarrow BK_{\mathsf{C}}$ will also be denoted Index and called the* index map.

Remark 2.4. *We relate the index map to boundary maps in algebraic K-theory in* [BGW14a, Thm. 3.25].

This property will be stated in Theorem 2.13 below, after having discussed algebraic K-theory more thoroughly in Subsection 2.3.

For every elementary Tate object V, we obtain from

$$B\,\mathrm{Aut}(V) \longrightarrow \mathsf{Tate}^{el}(\mathsf{C})^\times \xrightarrow{\mathsf{Index}} BK_{\mathsf{C}},$$

a map

$$\mathrm{Aut}(V) \longrightarrow K_{\mathsf{C}},$$

by applying the loop space functor Ω. By construction, it is a map of E_1-objects (i.e. homotopy coherent associative monoids). The E_1-structure in question stems from the concatenation of loops. In order to link our treatment to those appearing in the literature, it is important to give an algebraic treatment of this monoidal structure. We discuss this in detail in [BGW14a, Sect. 4].

One can also define a simplicial groupoid $Gr_\bullet(\mathsf{C})$, which classifies arbitrary tuples of lattices $(V; L_0, \ldots, L_k)$ in an elementary Tate object V. Subsection 4.3 of [BGW14a] outlines the construction of a the vertical arrow below, fitting into a commutative triangle

$$\begin{tikzcd} Gr^{\leq}_\bullet(\mathsf{C}) \arrow[r] \arrow[dr] & Gr_\bullet(\mathsf{C}) \arrow[d] \\ & K_{S_\bullet \mathsf{C}}. \end{tikzcd}$$

Considering 1-simplices, this defines a formal difference operation

$$Gr(V) \times Gr(V) \longrightarrow K_{\mathsf{C}},$$

which assigns to a pair of lattices (L_0, L_1) a point of K_{C}. We think of this construction as means of measuring the difference of L_0 and L_1 in a K-theoretic sense. For every lattice $M \supset L_0, L_1$ this difference is equivalent to $M/L_0 - M/L_1$.

If we fix a lattice L for every elementary Tate object V, we obtain a map $\mathrm{Aut}(V) \longrightarrow Gr_1(V)$, sending g to $(g^{-1}L, L)$. By means of the formal difference construction, i.e. by choosing a lattice N containing L and $g^{-1}L$ as common sub-lattices, we obtain an explicit description of the E_1-map $\mathrm{Aut}(V) \longrightarrow K_{\mathsf{C}}$.

Remark 2.5. *In* [BGW14a, Thm. 3.19] *we give an inductive proof for the existence of a compatible system of auxiliary choices of enveloping lattices, which can be organized into a simplicial map*

$$N_\bullet \,\mathsf{Tate}^{el}(\mathsf{C})^\times \longrightarrow d(N_\bullet \,\mathrm{Ex}^1\, Gr^{\leq}(\mathsf{C})),$$

where Ex^1 *denotes Kan's barycentric subdivision functor, and* d *denotes the diagonal simplicial set of a bisimplicial set.*

2.3. **Algebraic K-theory and Spectra.**

2.3.1. *Stable ∞-Categories.* We refer the reader to [Lura, Ch. 1] for a more detailed account. Every ∞-category C has an associated homotopy category $Ho(\mathsf{C})$, where the set of morphisms is defined to be the set of connected components

$$\mathsf{Hom}_{Ho(\mathsf{C})}(X, Y) = \pi_0 \,\mathsf{Hom}_{\mathsf{C}}(X, Y).$$

A *stable ∞-category* has a natural triangulated structure on its homotopy category. Examples include the stable ∞-category of *spectra*, and other enhancements of triangulated categories (for example pre-triangulated dg-categories).

By definition, a stable ∞-category C is pointed, i.e. there exists an initial and final object $\bullet$. Moreover, we assume the existence of finite limits and colimits, as well as that a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

is a pullback if and only if it is a pushout. The endofunctors $\Sigma\colon \mathsf{C} \longrightarrow \mathsf{C}$,

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \Sigma X, \end{tikzcd}$$

and $\Omega\colon \mathsf{C} \longrightarrow \mathsf{C}$,

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & X, \end{tikzcd}$$

are defined by virtue of the cocartesian, respectively cartesian squares above. As a consequence of the definition of a stable ∞-category, Σ and Ω are inverse equivalences. The induced functors on the homotopy category $Ho(\mathsf{C})$ give rise to the translation functors of the triangulated structure of $Ho(\mathsf{C})$. The distinguished triangles are the images of bicartesian squares

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ \bullet \arrow[r] & Z. \end{tikzcd}$$

Definition 2.6. *We denote the ∞-category of stable ∞-categories by* $\mathsf{Cat}_{\infty,\mathrm{st}}$.

A fundamental example is the stable ∞-category Sp of *spectra*, which is defined to be the limit

$$\mathsf{Sp} := \varprojlim[\mathsf{Space} \xleftarrow{\Omega} \mathsf{Space} \xleftarrow{\Omega} \cdots].$$

Every space X gives rise to a spectrum, denoted by $\Sigma^\infty X$. The infinite suspension functor

$$\Sigma^\infty \colon \mathsf{Space} \longrightarrow \mathsf{Sp}$$

has a right adjoint

$$\Omega^\infty \colon \mathsf{Sp} \longrightarrow \mathsf{Space}\,.$$

The latter functor is equivalent to the projection to the first component

$$\varprojlim[\mathsf{Space} \xleftarrow{\Omega} \mathsf{Space} \xleftarrow{\Omega} \cdots] \longrightarrow \mathsf{Space}\,.$$

There is an array of functors to the category of abelian groups

$$(\pi_i)_{i\in\mathbb{Z}} \colon \mathsf{Sp} \longrightarrow \mathrm{Ab},$$

inducing a t-structure on Sp with heart $\mathsf{Sp}^\heartsuit = \{X \in \mathsf{Sp}\,|\,\pi_i(X) = 0 \text{ for } i \neq 0\} \cong \mathrm{Ab}$.

The sub-category $\mathsf{Sp}_{[0,1]} = \{X \in \mathsf{Sp}\,|\,\pi_i(X) = 0 \text{ for } i \neq 0, 1\}$ is equivalent to the 2-category of *Picard groupoids*. More generally, the ∞-category of connective spectra $\mathsf{Sp}_{\geq} = \{X \in \mathsf{Sp}\,|\,\pi_i(X) = 0 \text{ for } i \leq -1\}$ is equivalent to the ∞-category of Segal's Γ-spaces (i.e. Picard ∞-groupoids, or equivalently, *infinite loop spaces*).

The behaviour of the ∞-category of spectra with respect to this t-structure reveals a remarkable similarity with the derived category $D(\mathbb{Z})$ of abelian groups. This time we have *homology groups*

$$H_i \colon D(\mathbb{Z}) \longrightarrow \mathrm{Ab},$$

inducing a t-structure on $D(\mathbb{Z})$. Again, the heart $D(\mathbb{Z})^\heartsuit$ is equivalent to the category of abelian groups. Chain complexes in $D(\mathbb{Z})_{[0,1]}$, i.e. those concentrated in degree 0 and 1, are, according to a theorem of Deligne, equivalent to *strictly commutative Picard groupoids*. The Dold-Kan correspondence asserts that objects in $D(\mathbb{Z})_{\geq 0}$ correspond to *simplicial abelian groups*.

It seems therefore appropriate to think of *spectra* as another generalization of abelian groups. The derived category of abelian groups serves a similar purpose, but working with spectra corresponds to only stipulating a *weak commutativity law*, which allows spectra to capture phenomena which could not be seen in the strict framework of chain complexes of abelian groups.

2.3.2. *A Blackbox Approach to K-Theory.* Instead of attempting to gain an explicit understanding of various K-groups, we view algebraic K-theory as a machine, which assigns, to an exact category or stable ∞-category C, its spectral shadow $\mathbb{K}_{\mathsf{C}}$. This machine sends exact functors $\mathsf{C} \longrightarrow \mathsf{D}$ to maps of spectra $\mathbb{K}_{\mathsf{C}} \longrightarrow \mathbb{K}_{\mathsf{D}}$, and preserves exact sequences. In this context, a short exact sequence of exact categories or stable ∞-categories refers to a localization

$$\mathsf{C} \hookrightarrow \mathsf{D} \twoheadrightarrow \mathsf{D}/\mathsf{C}$$

up to idempotent completion. An exact sequence of spectra is understood to be a bicartesian square

$$\begin{tikzcd}
\mathbb{K}_{\mathsf{C}} \arrow[r] \arrow[d] & \mathbb{K}_{\mathsf{D}} \arrow[d] \\
0 \arrow[r] & \mathbb{K}_{\mathsf{D}/\mathsf{C}},
\end{tikzcd}$$

which gives rise to a distinguished triangle in the homotopy category of spectra

$$\mathbb{K}_{\mathsf{C}} \longrightarrow \mathbb{K}_{\mathsf{D}} \longrightarrow \mathbb{K}_{\mathsf{D}/\mathsf{C}} \xrightarrow{\bullet}\,.$$

This triangle in turn induces a long exact sequence of homotopy groups

$$\cdots \longrightarrow K_i(\mathsf{C}) \longrightarrow K_i(\mathsf{D}) \longrightarrow K_i(\mathsf{D}/\mathsf{C}) \longrightarrow K_{i-1}(\mathsf{C}) \longrightarrow \cdots\,.$$

As a general principle, the K-theory of an exact (or stable ∞-) category admitting countable products or coproducts agrees with the zero spectrum. This is a spectral version of the *Eilenberg swindle*: if C admits countable coproducts, then, for every object X, we have an exact sequence

$$\bigoplus_{n\geq 1} X \hookrightarrow \bigoplus_{n\geq 0} X \twoheadrightarrow X,$$

which implies $X \simeq 0$ in K-theory.

Although those two properties of K-theory are convenient features, it will be necessary to compare algebraic K-theory with the original category, in order to be able to use it. Heuristically this is captured by the slogan that $\mathbb{K}_{\mathsf{C}}$ is a spectrum, where objects in C give rise to points, automorphisms of objects give rise to loops, and, for $n \geq 1$, commuting n-tuples of automorphisms in C give rise to elements of $K_n(\mathsf{C}) = \pi_n(\mathbb{K}_{\mathsf{C}})$.[2] This intuition is captured by the following observation.

Remark 2.7. *We denote by $\mathsf{C}^\times$ the $(\infty-)$groupoid of objects in C (i.e. we discard all non-isomorphisms). Recall that every $(\infty-)$groupoid can be viewed as a space (via the geometric realization of its nerve). There exists a canonical morphism of spaces*

$$\mathsf{C}^\times \longrightarrow \Omega^\infty \mathbb{K}_{\mathsf{C}},$$

and by the adjunction $\Sigma^\infty \dashv \Omega^\infty$, a morphism of spectra

$$\Sigma^\infty \mathsf{C}^\times \longrightarrow \mathbb{K}_{\mathsf{C}}\,.$$

Another gateway to concrete applications is present when C is the exact category of finitely generated projective R-modules $P_f(R)$. We denote by $B^{\mathbb{Z}}\,\mathbb{G}_m(R)$ the groupoid of $\mathbb{Z}$-graded invertible modules (M, m) (or line bundles, in a geometric notation). There is a natural graded tensor product, given by tensoring two invertible modules, and addition of integers. If one twists the symmetry constraint of tensoring two invertible modules M and N, by the sign $(-1)^{mn}$, one obtains the symmetric monoidal groupoid $\mathbb{B}^{\mathbb{Z}}\,\mathbb{G}_m(R)$. As we have seen, we can view $\mathbb{B}^{\mathbb{Z}}\,\mathbb{G}_m(R)$ as an element in $\mathsf{Sp}_{[0,1]}$. There exists a canonical morphism of spectra

$$\mathcal{K}_R = \tau_{\geq 0}\,\mathbb{K}_R \longrightarrow \mathbb{B}^{\mathbb{Z}}\,\mathbb{G}_m(R).$$

In particular, applying the functors π_0 and π_1 yields natural maps $K_0(R) \xrightarrow{\text{rank}} \mathbb{Z}$ and $K_1(R) \xrightarrow{\det} R^\times$.

2.3.3. *Connective Algebraic K-theory.* The proposition below captures the most important phenomena for K-theory of stable ∞-categories (cf. [BGT13]).

Proposition 2.8. *The functor of connective K-theory for stable ∞-categories*

$$\mathcal{K}_{-}\colon\ \mathsf{Cat}_{\infty,\mathrm{st}} \longrightarrow \mathsf{Sp}_{\geq}$$

satisfies the following properties.

1. *If C is a stable ∞-category admitting countable products (or coproducts), then $\mathcal{K}_{\mathsf{C}} \cong 0$.*
2. *The inclusion $\mathsf{C} \longrightarrow \mathsf{C}^{\mathrm{ic}}$ (where* ic *denotes idempotent completion) gives rise to a map of connective spectra $\mathcal{K}_{\mathsf{C}} \longrightarrow \mathcal{K}_{\mathsf{C}^{\mathrm{ic}}}$, inducing an isomorphism on π_i for $i \geq 1$, and a monomorphism on π_0.*
3. *An exact sequence $\mathsf{C} \hookrightarrow \mathsf{D} \twoheadrightarrow \mathsf{D}/\mathsf{C}$ of stable ∞-categories induces a fibre sequence*

$$\begin{tikzcd} \mathcal{K}_{\mathsf{C}} \arrow[r] \arrow[d] & \mathcal{K}_{\mathsf{D}} \arrow[d] \\ 0 \arrow[r] & \mathcal{K}_{\mathsf{D}/\mathsf{C}} \end{tikzcd}$$

in the ∞-category $\mathsf{Sp}_{\geq}$ of connective spectra.

[2]We will explain in detail how commuting n-tuples of automorphisms yield elements of $\pi_n(\mathbb{K}_{\mathsf{C}})$ in Definition 3.30.

Property (3) is often referred to as *proto-localization* (e.g. by [TT90]). The long exact fibration sequence for $\pi_\bullet$ yields a long exact sequence in non-negative degrees. The map $\pi_0(\mathcal{K}_{\mathsf{C}}) \longrightarrow \pi_0(\mathcal{K}_{\mathsf{C}})$ will not be surjective in general. This suggests the existence of negative K-groups, obtained by the homotopy groups of a non-connective K-theory spectrum.

2.3.4. *Non-Connective Algebraic K-Theory.* In the work of Blumberg–Gepner–Tabuada, the following properties were shown to be characteristic for non-connective K-theory (see [BGT13, Thm. 9.10]).

Proposition 2.9. *Non-connective algebraic K-theory is a functor*

$$\mathbb{K}_{-}\colon \mathsf{Cat}_{\infty,\mathrm{st}} \longrightarrow \mathsf{Sp}$$

satisfying the following properties.

1. *If C is a stable ∞-category admitting countable products (or coproducts), then $\mathbb{K}_{\mathsf{C}} \cong 0$.*
2. *The inclusion $\mathsf{C} \longrightarrow \mathsf{C}^{\mathrm{ic}}$ (where* ic *denotes idempotent completion) gives rise to an equivalence of spectra $\mathbb{K}_{\mathsf{C}} \xrightarrow{\cong} \mathbb{K}_{\mathsf{C}^{\mathrm{ic}}}$.*
3. *An exact sequence $\mathsf{C} \hookrightarrow \mathsf{D} \twoheadrightarrow \mathsf{D}/\mathsf{C}$ of stable ∞-categories induces a bi-cartesian square*

$$\begin{tikzcd} \mathbb{K}_{\mathsf{C}} \arrow[r] \arrow[d] & \mathbb{K}_{\mathsf{D}} \arrow[d] \\ 0 \arrow[r] & \mathbb{K}_{\mathsf{D}/\mathsf{C}} \end{tikzcd}$$

in the stable ∞-category Sp of spectra.

We say that non-connective K-theory $\mathbb{K}_{-}$ *completes* connective K-theory $\mathcal{K}_{-}$, referring to the canonical equivalence

$$\mathcal{K}_{\mathsf{C}^{\mathrm{ic}}} \cong \tau_{\geq 0}\,\mathbb{K}_{\mathsf{C}}\,.$$

Following Schlichting, we see how every connective theory, satisfying the axioms of Proposition 2.8, induces a non-connective K-theory, subject to the properties of Proposition 2.9 (see also [BGT13]). This requires the *suspension* of a stable ∞-category.

Definition 2.10. *We define the* suspension *of a stable ∞-category C as the stable ∞-category*

$$\mathcal{S}_\kappa(\mathsf{C}) = \mathsf{Ind}_\kappa \mathsf{C}/\mathsf{C},$$

where κ denotes an arbitrary infinite cardinal. Let $\mathsf{Calk}_\kappa(\mathsf{C})$ denote $\mathcal{S}_\kappa(\mathsf{C})^{\mathrm{ic}}$.

By definition, we have an exact sequence of stable ∞-categories

$$\mathsf{C} \hookrightarrow \mathsf{Ind}(\mathsf{C}) \twoheadrightarrow \mathcal{S}(\mathsf{C}).$$

Using the fact that $\mathsf{Ind}(C)$ admits countable coproducts, properties (1) and (3) of Proposition 2.8 imply that

$$\mathcal{K}_C \longrightarrow 0 \longrightarrow \mathcal{K}_{\mathcal{S}(\mathsf{C})}$$

is a fibre sequence of connective spectra. Since $\pi_0(0) = \pi_0(\mathcal{S}(\mathsf{C}))$, we know that it is actually a fibre-cofibre sequence of spectra. This allows us to identify $\mathcal{K}_C$ with $\Omega\,\mathcal{K}_{\mathcal{S}(\mathsf{C})}$. We define the non-connective completion $\mathbb{K}_{-}$ to be the functor

$$\varinjlim \Omega^n\,\mathcal{K}_{\mathsf{Calk}(\mathsf{C})}\,. \tag{6}$$

Definition 2.11. *Let C be an idempotent complete exact category. We have a well-defined dg-category $\mathsf{Ch}^{\mathrm{b}}(\mathsf{C})$ of bounded chain complexes in C. We denote by $\mathsf{Ch}^{\mathrm{b}}_{\mathrm{ac}}(\mathsf{C})$ the full sub-category of acyclic complexes. The stable ∞-category* $\mathrm{Perf}(\mathsf{C})$ *is defined to be the dg-nerve (see* [Lura, Sect. 1.3.1]*) of the dg-quotient $\mathsf{Ch}^{\mathrm{b}}(\mathsf{C})/\,\mathsf{Ch}^{\mathrm{b}}_{\mathrm{ac}}(\mathsf{C})$. Since the latter is a pre-triangulated dg-category (see* [Kel99, Sect. 2]*),* $\mathrm{Perf}(\mathsf{C})$ *is stable.*

The lemma below follows from the discussion in [BGT13, sect. 9.1] and [Sch06, sect. 6.2]

Lemma 2.12. *Let* C *be an exact category. The non-connective* K*-theory of* C*, in the sense of Schlichting* [Sch06]*, agrees with the non-connective* K*-theory of the stable* ∞*-category* $\mathrm{Perf}(\mathsf{C})$ *in the sense of Blumberg–Gepner–Tabuada* [BGT13].

2.4. **Delooping Constructions.**

2.4.1. *For Exact Categories.* Schlichting developed a Localization Theorem for exact categories in [Sch04], which states that, for every left (respectively right) s-filtering inclusion of exact categories

$$\mathsf{C} \hookrightarrow \mathsf{D},$$

the quotient category D/C carries a natural structure as an exact category. Further, if C is idempotent complete, then by applying K-theory, we obtain a bicartesian square of spectra

$$\begin{tikzcd} \mathbb{K}_{\mathsf{C}} \arrow[r] \arrow[d] & \mathbb{K}_{\mathsf{D}} \arrow[d] \\ 0 \arrow[r] & \mathbb{K}_{\mathsf{D}/\mathsf{C}}. \end{tikzcd}$$

Schlichting observed that if one chooses D such that $\mathbb{K}_{\mathsf{D}} \cong 0$, then the boundary morphism of this square gives an equivalence

$$\partial\colon\ \mathbb{K}_{\mathsf{D}/\mathsf{C}} \cong \Sigma\,\mathbb{K}_{\mathsf{C}}.$$

The Eilenberg swindle guarantees that $\mathbb{K}_{\mathsf{Ind}^a(\mathsf{C})} \cong 0$ for every exact category C. Thus, we see that, for C idempotent complete, we have a canonical delooping

$$\mathbb{K}_{\mathsf{Ind}^a(\mathsf{C})/\mathsf{C}} \cong \Sigma\,\mathbb{K}_{\mathsf{C}}.$$

Using similar techniques, Saito establishes an abstract equivalence

$$\mathbb{K}_{\mathsf{Tate}(\mathsf{C})} \cong \Sigma\,\mathbb{K}_{\mathsf{C}}$$

in [Sai12]. In fact, this equivalence can be constructed as the composition

$$\mathbb{K}_{\mathsf{Tate}(\mathsf{C})} \cong \mathbb{K}_{\mathsf{Tate}^{el}(\mathsf{C})} \cong \mathbb{K}_{\mathsf{Tate}^{el}(\mathsf{C})/\mathsf{Pro}^a(\mathsf{C})} \cong \mathbb{K}_{\mathsf{Ind}^a(\mathsf{C})/\mathsf{C}} \cong \Sigma\,\mathbb{K}_{\mathsf{C}}.$$

The first equivalence follows from the cofinal invariance of non-connective K-theory (i.e. (2) of Proposition 2.9). The second map is an equivalence as a corollary of the aforementioned Localization Theorem, and the third equivalence exists already on the level of exact categories (e.g. [BGW14b, Prop. 5.28]).

The index map of Definition 2.3 is an explicit description of these boundary maps. See [BGW14a, Thm. 3.25] for proof.

Theorem 2.13. *Let* C *be an idempotent complete exact category. The equivalence* $\mathsf{Tate}^{el}(\mathsf{C})/\mathsf{Pro}(\mathsf{C}) \cong \mathsf{Ind}^a(\mathsf{C})/\mathsf{C}$ *(see* [BGW14b, Prop. 5.28]*) induces a commutative diagram*

$$\begin{tikzcd} \Sigma^\infty \mathsf{Tate}^{el}(\mathsf{C})^\times \arrow[r, "\mathsf{Index}"] \arrow[d] & \Sigma\,\mathbb{K}_{\mathsf{C}} \\ \mathbb{K}_{\mathsf{Ind}(\mathsf{C})/\mathsf{C}} \arrow[ur, "-\partial"'] & \end{tikzcd}$$

of boundary maps.

This theorem motivates the following definition of the *non-connective* index map.

Definition 2.14. *For an idempotent complete exact category* C*, we define the map* $\mathsf{Index}\colon\ \mathbb{K}_{\mathsf{Tate}(\mathsf{C})} \longrightarrow \Sigma\,\mathbb{K}_{\mathsf{C}}$ *as the composition*

$$\begin{tikzcd} \mathbb{K}_{\mathsf{Tate}(\mathsf{C})} \arrow[r, "\mathsf{Index}"] \arrow[d] & \Sigma\,\mathbb{K}_{\mathsf{C}} \\ \mathbb{K}_{\mathsf{Ind}(\mathsf{C})/\mathsf{C}}. \arrow[ur, "-\partial"'] & \end{tikzcd}$$

2.4.2. *Suspension and Calkin Objects for Stable ∞-Categories.* Let C be a stable ∞-category, and κ an infinite cardinal. Recall Definition 2.10, which defines the *suspension* $\mathcal{S}_\kappa(\mathsf{C})$ as the localization

$$\mathcal{S}_\kappa(\mathsf{C}) = \mathsf{Ind}_\kappa(\mathsf{C})/\mathsf{C},$$

and which defines the ∞-category of Calkin objects $\mathsf{Calk}(\mathsf{C})$ as the idempotent completion of the suspension.

Since non-connective K-theory cannot distinguish between a category and its idempotent completion (see (2) of Proposition 2.9):

$$\mathbb{K}_{\mathsf{Calk}_\kappa(\mathsf{C})} \cong \mathbb{K}_{\mathcal{S}_\kappa(\mathsf{C})} \cong \Sigma\, \mathbb{K}_{\mathsf{C}},$$

we will often omit the cardinal κ from our notation. Following Schlichting [Sch04], Blumberg–Gepner–Tabuada [BGT13] observed the following delooping property for K-theory introduced in (6).

Proposition 2.15. *The boundary map ∂ of the localization sequence of the exact sequence*

$$\mathsf{C} \hookrightarrow \mathsf{Ind}(\mathsf{C}) \twoheadrightarrow \mathcal{S}(\mathsf{C})$$

of stable ∞-categories, induces an equivalence of non-connective K-theory spectra $\partial\colon\ \mathbb{K}_{\mathsf{Calk}(\mathsf{C})} \cong \Sigma\, \mathbb{K}_{\mathsf{C}}$.

This result serves as a motivation to call $\mathcal{S}(\mathsf{C})$ the suspension of C. Recall that the suspension of a topological space X is formed by embedding X into the cone CX, which is *contractible*. The resulting homotopy cofibre, obtained by taking the quotient space, is one possible incarnation of the suspension. By analogy, Schlichting embeds a category C into an ambient K-contractible category $\mathsf{Ind}(\mathsf{C})$, and takes the quotient to obtain the categorical suspension. A second possibility is to construct the suspension of X by glueing a second copy C^-X to the cone CX along the common subspace X. Since C^-X is contractible, this yields a homotopy equivalent space. Categorically this is analogous to pasting the K-contractible categories $\mathsf{Ind}^a(\mathsf{C})$ and $\mathsf{Pro}^a(\mathsf{C})$ along the common subcategory C. This is the underlying idea of Saito's delooping statement. For later use, we record the following naturality property. The proof is an exercise in unravelling definitions.

Lemma 2.16 (Naturality)**.** *For every idempotent complete exact category C, there exists a commutative diagram*

$$\begin{tikzcd}
\mathbb{K}_{\mathsf{C}} \arrow[r, "\cong"] \arrow[d] & \mathbb{K}_{\mathrm{Perf}(\mathsf{C})} \arrow[d] \\
\Sigma\, \mathbb{K}_{\mathsf{Ind}^a(\mathsf{C})/\mathsf{C}} \arrow[r, "\cong"] & \Sigma\, \mathbb{K}_{\mathsf{Calk}(\mathrm{Perf}(\mathsf{C}))}
\end{tikzcd}$$

of spectra, where the horizontal maps $\mathbb{K}_{\mathsf{C}} \cong \mathbb{K}_{\mathrm{Perf}(\mathsf{C})}$ are the equivalences stipulated by Lemma 2.12.

3. Spectral Extensions and Higher Commutators

In this section, we adopt the conventions that all groups are discrete. Group operations are also written multiplicatively, unless stated otherwise. In this section, the term *space* refers to a *pointed* space.

3.1. **Classical Theory.**

3.1.1. *Central Extensions.* Let A be an abelian group. A *central extension of G by A*, denoted e, is a short exact sequence

$$1 \longrightarrow A \xrightarrow{\iota} E \xrightarrow{p} G \longrightarrow 1, \tag{7}$$

such that $\iota(A) \subset Z(E)$, where $Z(E) \subset E$ denotes the centre of E.

Definition 3.1. *We denote by $P_n(G)$ the set*

$$\{(g_1, \ldots, g_n) \in G^n \mid g_i g_j = g_j g_i \ \forall\ 1 \le i, j \le n\}$$

of n-tuples of pairwise commuting elements.

Given $(f,g) \in P_2(G)$, let $\widetilde{f}$, $\widetilde{g}$ be elements in $p^{-1}(f)$, respectively $p^{-1}(g)$. Since $p(\widetilde{f}\widetilde{g}\widetilde{f}^{-1}\widetilde{g}^{-1}) = 1$, we see that the commutator $[\widetilde{f},\widetilde{g}] = \widetilde{f}\widetilde{g}\widetilde{f}^{-1}\widetilde{g}^{-1}$ defines an element in A. By a simple computation, one sees that this element is independent of the choice of liftings.

Definition 3.2. *Let e be a central extension of G by A. We denote by $\star_e \colon P_2(G) \longrightarrow A$ the function $(f,g) \mapsto \iota^{-1}[\widetilde{f},\widetilde{g}]$.*

A short exercise shows that $\star$ is bimultiplicative and anti-symmetric.

Lemma 3.3. *For $(g_1,h),(g_2,h) \in P_2(G)$, respectively $(g,h) \in P_2(G)$ we have the following relations:*

(a) $(g_1 \star_e h) \cdot (g_2 \star_e h) = g_1 g_2 \star_e h$,

(b) $(g \star_e h)^{-1} = h \star_e g$.

A central extension e as in (7) corresponds to a monoidal map from G to the classifying space BA. To see this directly, one observes that every fibre $p^{-1}(g) \subset E$ has the structure of an A-torsor. Moreover, we have a natural isomorphism $p^{-1}(gh) \cong p^{-1}(g) \otimes_A p^{-1}(h)$ for every pair $(g,h) \in G^2$. Thus, (7) gives rise to a map of monoidal groupoids

$$\phi \colon G \longrightarrow BA, \tag{8}$$

where G is viewed as a discrete groupoid with monoidal structure given by the group operation, and BA denotes the classifying groupoid of A-torsors. The following interpretation of the commutator pairing is well-known.

Lemma 3.4. *For $(f,g) \in P_2(G)$ we have that $f \star g$ corresponds to the automorphism in BA obtained from the following chain of morphisms*

$$\phi(fg) \cong \phi(f)\phi(g) \cong \phi(g)\phi(f) \cong \phi(gf) = \phi(fg).$$

Proof. Choosing lifts $\widetilde{f}$ of f and $\widetilde{g}$ of g, we can express the torsors $\phi(f)$ as $A \cdot \widetilde{f}$ and $\phi(g)$ as $A \cdot \widetilde{g}$. We can also write

$$\phi(fg) \cong A \cdot \widetilde{f}\widetilde{g} \cong \phi(f) \otimes_A \phi(g).$$

The symmetry constraint of $\otimes_A$ induces an isomorphism with $A \cdot \widetilde{g}\widetilde{f}$, which sends $\widetilde{f}\widetilde{g}$ to $[\widetilde{f},\widetilde{g}]\widetilde{g}\widetilde{f}$.

Using the identification $\phi(g)\phi(f) \cong \phi(gf) = \phi(fg)$ the element $\widetilde{g}\widetilde{f}$ is sent to $\widetilde{f}\widetilde{g}$. We conclude that the resulting automorphism of the torsor $\phi(fg) \cong A \cdot \widetilde{f}\widetilde{g}$ sends the element $\widetilde{f}\widetilde{g}$ to $[\widetilde{f},\widetilde{g}]\widetilde{f}\widetilde{g}$. Therefore, it corresponds to the commutator pairing $f \star g$. □

3.1.2. *Cohomological Reformulation.* The map (8) is the looping of the map of classifying spaces

$$e \colon BG \longrightarrow B^2 A.$$

Since the target is equivalent to an Eilenberg–Mac Lane space $B^2 A \cong K(A,2)$, homotopy classes of such maps agree with $H^2(BG,A) = H^2_{grp}(G,A)$. We denote the resulting element in this cohomology group by $[e]$.

If G is an abelian group, then the group homology $H_*(BG,\mathbb{Z}) = H_*^{grp}(G,\mathbb{Z})$ carries a natural graded commutative ring structure. Topologically this follows from BG inheriting a group structure from the commutative group G, endowing it with the structure of an H-group. Algebraically, this fact can be explained in terms of the *shuffle product* on the normalized bar complex. In the remark below we recall its definition.

Remark 3.5. *Recall that the $\mathbb{Z}G$-module B_k is defined to be the free module on symbols $(g_1|\dots|g_k)$, where the g_i are pairwise distinct elements of the group G. Using that G is abelian, we define*

$$(g_1|\dots|g_k) \circ (g_{k+1}|\dots|g_{k+l}) = \sum_\sigma (-1)^\sigma (g_{\sigma^{-1}(1)}|\dots|g_{\sigma^{-1}(k+l)}),$$

where σ runs over all permutations of $\{1,\dots,k+l\}$ satisfying $\sigma(1) \leq \dots \leq \sigma(k)$ and $\sigma(k+1) \leq \dots \leq \sigma(k+l)$ (so-called shuffles*). Extending $\mathbb{Z}G$-linearly, the shuffle product endows $\bigoplus_k B_k$ with the structure of a commutative dg-algebra.*

This graded commutative ring structure brings us to the following definition.

Definition 3.6. *Let G be an arbitrary group. Given $(g_1, \ldots, g_n) \in P_n(G)$, we denote by $\phi\colon \mathbb{Z}^n \longrightarrow G$ the corresponding morphism of groups, sending the standard vector e_i to g_i. Let c denote $(e_1 \circ \cdots \circ e_n)$ as in Remark 3.5. We set $(g_1 \circ \cdots \circ g_n) := \phi_*(c)$.*

The class in $H_2^{grp}(G, \mathbb{Z})$ corresponding to the cycle $(f \circ g)$ should be understood as an *abstract commutator*. A pair (f, g) of commuting elements induces a map $\mathbb{T}^2 = B\mathbb{Z}^2 \longrightarrow BG$. Topologically speaking, the cycle $(f \circ g)$ is obtained by pushforward of the fundamental class of the torus $B\mathbb{Z}^2$ to BG.

Lemma 3.7. *Let $\langle -, - \rangle$ denote the natural pairing between group cohomology and homology. Given a central extension e of G by A, corresponding to the class $[e] \in H^2_{grp}(G, A)$, we have for all $(f, g) \in P_2(G)$ the identity*

$$f \star_e g = \langle [e], (f \circ g) \rangle.$$

The following definition illustrates the flexibility of the cohomological viewpoint on commutators.

Definition 3.8. *A* higher central extension *of G by $B^k A = K(A, k)$ is an element $[e]$ of $H^{k+2}_{grp}(G, A)$. Given $(g_1, \ldots, g_{k+2}) \in P_{k+2}(G)$ we define*

$$g_1 \star_e \cdots \star_e g_{k+2} = \langle [e], (g_1 \circ \cdots \circ g_{k+2}) \rangle.$$

In the following subsection we will formally generalize this definition to include central extensions by arbitrary spectra, not just those of Eilenberg–Mac Lane type.

3.2. **Spectral Extensions.**

3.2.1. *Generalized Group Cohomology.* For every spectrum $\mathbb{E}$, we have an associated *generalized cohomology theory* denoted by $H^i(-, \mathbb{E})$. We define generalized group cohomology to be $H^i_{grp}(G, \mathbb{E}) = H^i(BG, \mathbb{E})$.

Definition 3.9. *A* spectral extension *of G by $\mathbb{E}$ is a map $e\colon \Sigma^\infty BG \longrightarrow \Sigma^2\mathbb{E}$. The induced class in $H^2_{grp}(G, \mathbb{E})$ will be denoted by $[e]$.*

Every abelian group A can be viewed as a spectrum HA, by means of the Eilenberg–Mac Lane construction. In particular we have $\Omega^\infty \Sigma^k HA = B^k A$.

A higher central extension of G by $B^k A$ in the sense of Definition 3.8, is given by an element of $H^{k+2}_{grp}(G, A) = H^2_{grp}(G, \Sigma^k HA)$. By the discussion above, we can therefore say that a higher central extension of G by $B^k A$ is the same thing as a spectral extension of G by the k-fold suspension spectrum $\Sigma^k HA$.

Remark 3.10. *The definition of a spectral extension given in Definition 3.9 introduces only the cocycle (up to equivalence) of what should be a* central extension by a spectrum. *Without doubt it would be possible to give a definition along the lines of Paragraph 3.1.1. However, spelling out such a definition would certainly be more laboursome than the shortcut used in Definition 3.9, which is exactly the viewpoint we need to study higher commutators in the next paragraph. For this reason we refrain from developing a theory of spectral extensions in the classical spirit. Still, when inclined to do so, we will allude to a map*

$$e\colon \Sigma^\infty BG \longrightarrow \Sigma^2\mathbb{E}$$

as classifying *a spectral extension of G by $\mathbb{E}$, although strictly speaking we have not explained an alternative way to think of spectral extensions.*

3.2.2. *Higher Commutators.* In order to define an analogue of the homology cycles $(g_1 \circ \cdots \circ g_n)$ of Definition 3.6, we remind the reader of an elementary result in homotopy theory.

Lemma 3.11. *Let X, Y be spaces. There is a canonical homotopy equivalence*

$$\Sigma(X \times Y) \cong \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y).$$

In particular, there exists a canonical sum decomposition of spectra

$$\Sigma^\infty(X \times Y) \cong \Sigma^\infty X \oplus \Sigma^\infty Y \oplus \Sigma^\infty(X \wedge Y).$$

Applying this lemma recursively to the product $\mathbb{T}^n = (\mathbb{S}^1)^n$ we obtain a canonical element in the stable homotopy group $\pi_n^s(\mathbb{T}^n)$.

Definition 3.12. *We denote the canonical splitting of the morphism $\Sigma^\infty\mathbb{T}^n \longrightarrow \Sigma^\infty\mathbb{S}^n$ by c_n. Given $(g_1, \dots, g_n) \in P_n(G)$, let $\phi\colon \mathbb{Z}^n \longrightarrow G$ be the resulting map sending the standard vector $e_i \mapsto g_i$. We define*

$$(g_1 \sharp \dots \sharp g_n)\colon \Sigma^\infty\mathbb{S}^n \longrightarrow \Sigma^\infty BG$$

to be $\Sigma^\infty B\phi \circ c_n$.

Remark 3.13. *Let $h\colon \pi_*^s \longrightarrow H_*$ denote the Hurewicz homomorphism. We have*

$$(g_1 \circ \cdots \circ g_n) = h(g_1 \sharp \dots \sharp g_n).$$

We are now prepared to define higher commutators.

Definition 3.14. *Let e be a spectral extension of G by $\mathbb{E}$, and $(g_1, \dots, g_n) \in P_n(G)$. We define*

$$g_1 \star_e \cdots \star_e g_n \colon \Sigma^\infty\mathbb{S}^n \longrightarrow \Sigma^2\mathbb{E}$$

to be the composition of the map $\Sigma^\infty\mathbb{S}^n \longrightarrow \Sigma^\infty B\mathbb{Z}^n$ (Definition 3.12) with the map $\Sigma^\infty B\mathbb{Z}^n \xrightarrow{\varphi} \Sigma^\infty BG \xrightarrow{e} \mathbb{E}$. The resulting element of $\pi_{n-2}(\mathbb{E})$ will also be denoted by $g_1 \star_e \cdots \star_e g_n$.

Lemma 3.15. *Let e be a spectral extension of G by $\mathbb{E}$. Given $(g, g_2, \dots, g_n), (h, g_2, \dots, g_n) \in P_n(G)$, respectively $(g_1, \dots, g_n) \in P_n(G)$, the following identities hold in $\pi_{n-2}\mathbb{E}$:*

(a) $(g \star_e \cdots \star_e g_n) \cdot (h \star_e \cdots \star_e g_n) = (gh \star_e \cdots \star_e g_n)$,

(b) $(g_{\sigma(1)} \star_e \cdots \star_e g_{\sigma(n)}) = (-1)^{|\sigma|}(g_1 \star_e \cdots \star_e g_n)$, *where σ denotes a permutation.*

Proof. The two identities are consequences of the formal behaviour of the abstract higher commutator of Definition 3.12. Equation (a) follows from the compatibility of the stable splitting of Lemma 3.11 with the H-cogroup structure on $\mathbb{S}^1$.

Equation (b) amounts to the observation that a permutation of the factors of the splitting $\Sigma^\infty\mathbb{S}^n \longrightarrow \Sigma^\infty(\mathbb{S}^1)^n$ induces a change of sign by $|\sigma|$. □

In the statement above, if $\mathbb{E}$ is chosen to be the spectrum $\Sigma^n HA$, where A is an abelian group, the assertions above would follow from the formal properties of the shuffle product on group homology (Remark 3.5).

3.2.3. *Spectral Extensions of Groupoids.* Most examples of spectral extensions arising in algebraic K-theory (e.g. cf. Section 3.3) are not initially given by base point preserving maps $BG \longrightarrow \Omega^\infty\Sigma^2\mathbb{E}$. Instead, one often has a map $(BG)_+ \longrightarrow \Omega^\infty\Sigma^2\mathbb{E}$. The space $(BG)_+$ represents a groupoid; we therefore develop the theory of spectral extensions of groupoids in this paragraph. The equivalence between spaces and ∞-groupoids (cf. Section 2.1.1) motivates the following.

Definition 3.16. *Let X be a space and $\mathbb{E}$ a spectrum. A map $e\colon \Sigma^\infty X \longrightarrow \Sigma^2\mathbb{E}$ is called a* spectral extension *of X by $\mathbb{E}$.*

The basic properties of spectra yield the next observation.

Remark 3.17. *Since $\Sigma^\infty X$ is a connective spectrum, the morphism $\Sigma^\infty X \longrightarrow \Sigma^2\mathbb{E}$ factors through the connective cover $\Sigma^2\tau_{\geq -2}\mathbb{E}$. In particular, we see that there is an equivalence between spectral extensions e of the space X by $\mathbb{E}$ and spectral extensions $\tau(e)$ of X by $\tau_{\geq -2}\mathbb{E}$.*

As a first step in defining higher commutators in this context, we need to reflect on the generalization of the notion of tuples of pairwise commuting elements in a group.

Definition 3.18. *Let X be a space, and $x \in X$ a point, which may differ from the base point of X. We define $P_n(X,x)$ to be the space of maps $\mathbb{T}^n_+ = (\mathbb{S}^1_+)^{\wedge n} \longrightarrow X$, sending the base point x_0 of $\mathbb{T}^n$ to x.*

If X is the nerve of a groupoid C, then $P_n(X,x)$ agrees indeed with the set of n-tuples of pairwise commuting elements in $\pi_1(X,x) = Aut_{\mathsf{C}}(x)$.

Definition 3.19. *Let X be a space, $x \in X$ and $(g_1,\dots,g_n) \in P_n(X,x)$. Then we define, for every spectral extension $e\colon \Sigma^\infty X \longrightarrow \Sigma^2\mathbb{E}$, the higher commutator*

$$(g_1 \star_e \cdots \star_e g_n)\colon \Sigma^\infty(\mathbb{S}^n)_+ \longrightarrow \Sigma^\infty(\mathbb{T}^n)_+ \longrightarrow \Sigma^2\mathbb{E}$$

just as in Definition 3.14. The resulting element of $\pi_{n-2}(\mathbb{E})$ will be denoted by the same notation.

Continuing Remark 3.17, we observe that the formalism of higher commutators is not sensitive to replacing the spectrum $\mathbb{E}$ by $\tau_{\geq -2}\mathbb{E}$.

Remark 3.20. *For X a space, $x \in X$ a point, $(g_1,\dots,g_n) \in P_n(X,x)$, and $e\colon X \longrightarrow \Sigma^2\mathbb{E}$ a spectral extension, we have*

$$(g_1 \star_e \cdots \star_e g_n) \simeq (g_1 \star_{\tau(e)} \cdots \star_{\tau(e)} g_n) \in \pi_{n-2}(\mathbb{E}).$$

We also have the following definition.

Definition 3.21. *Let X be a space, $x \in X$ and $(g_1,\dots,g_n) \in P_n(X,x)$. We denote by $e[x]$ the map adjoint to $B\mathbb{Z}^n \longrightarrow \Omega^\infty\mathbb{E}$, which is obtained by composing the adjoint $X \longrightarrow \Omega^\infty\mathbb{E}$ of e with the subtraction map $-\ominus e(x_0)\colon \Omega^\infty\mathbb{E} \longrightarrow \Omega^\infty\mathbb{E}$ coming from the E_∞-structure on $\Omega^\infty\mathbb{E}$.*

This construction allows us to conclude that the treatment of higher commutators in the context of spectral extensions of groups is general enough to imply results on higher commutators for spectral extensions of ∞-groupoids.

Lemma 3.22. *Using the same assumptions as in Definition 3.21 we have*

$$(g_1 \star_e \cdots \star_e g_n) = (g_1 \star_{e[x]} \cdots \star_{e[x]} g_n) \in \pi_{n-2}(\mathbb{E}).$$

3.2.4. *A Picture.* We will describe an alternative construction of the (higher) commutator in $\pi_0(\mathbb{E})$ associated to a spectral extension $\Sigma^\infty\mathbb{T}^2 \longrightarrow \Sigma^2\mathbb{E}$. To explain the general idea, we assume that $\mathbb{E}$ is connective, i.e. corresponds to a group-like E_∞-object $\mathcal{E}$ in the category of spaces. In this case, $\Sigma^2\mathbb{E}$ corresponds to the 2-fold classifying space $B^2\mathcal{E}$, and a map $\mathbb{T}^2_+ \longrightarrow B^2\mathcal{E}$ represents a $B\mathcal{E}$-torsor (hence an $\mathcal{E}$-gerbe on $\mathbb{T}^2_+$). We will describe this $B\mathcal{E}$-torsor by patching trivial torsors.

The complement of a point in $\mathbb{T}^2$ is homotopy equivalent to the wedge of two circles $\mathbb{S}^1 \vee \mathbb{S}^1$. Hence, $\mathbb{T}^2$ is the union of a tubular neighbourhood of a bouquet of two circles (i.e. $\mathbb{T}^2 \setminus \bullet$) and a disc $\mathbb{D}$, intersecting in an annulus $\mathbb{D}^\bullet$.

By passing to the respective deformation retracts, we see that

(9)
$$\begin{tikzcd} \mathbb{S}^1 \arrow[r, "j"] \arrow[d, "{[f,g]}"'] & \bullet \arrow[d] \\ \mathbb{S}^1 \vee \mathbb{S}^1 \arrow[r] & \mathbb{T}^2 \end{tikzcd}$$

is a pushout in the ∞-category of spaces. The map $[f,g]\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1 \vee \mathbb{S}^1$ is given by the abstract commutator, i.e. the loop given by the word $fgf^{-1}g^{-1}$. Here f and g denote the identity maps $\mathbb{S}^1 \longrightarrow \mathbb{S}^1$ on the first, respectively the second copy of $\mathbb{S}^1$ in $\mathbb{S}^1 \vee \mathbb{S}^1$.

The datum of a $B\mathcal{E}$-torsor $\mathcal{T}$ corresponds to a $B\mathcal{E}$-torsor $\mathcal{T}_1$ on $\mathbb{T}^2 \setminus \bullet$, a $B\mathcal{E}$-torsor $\mathcal{T}_2$ on $\mathbb{D}$, and an isomorphism of torsors $[f,g]^*\mathcal{T}_1 \cong j^*\mathcal{T}_2$. Since $\mathbb{D} \cong \bullet$, we have an equivalence $\mathcal{T}_2 \cong B\mathcal{E}$ with the trivial torsor. Similarly

$$[f,g]^*\mathcal{T}_1 \cong g^*\mathcal{T}_1^{-1} \otimes f^*\mathcal{T}_1^{-1} \otimes g^*\mathcal{T}_1 \otimes f^*\mathcal{T}_1 \cong B\mathcal{E}$$

is canonically trivialised, as a consequence of the E_∞-structure on the classifying space $B^2\mathcal{E}$. The descent datum is therefore given by an automorphism of the trivial $B\mathcal{E}$-torsor over $\mathbb{S}^1$, i.e. a map $\mathbb{S}^1 \longrightarrow B\mathcal{E}$. Since $\Omega B\mathcal{E} \cong \mathcal{E}$, we obtain a canonical element of $\pi_0\mathcal{E}$, associated to the spectral extension.

Remark 3.23. *As a reality check, one obtains that the higher commutator associated to a map $\mathbb{T}^2 \longrightarrow B^2A$, which corresponds to a central extension $\phi\colon \mathbb{Z}^2 \longrightarrow BA$, i.e.*

$$0 \longrightarrow A \longrightarrow E \longrightarrow \mathbb{Z}^2 \longrightarrow 0,$$

is indeed equal to the classical commutator pairing. To see this, note that the patching function for the resulting BA-torsor is given by the automorphism

$$\mathbf{0} \cong \phi(g) \oplus \phi(f) \oplus -\phi(g) \oplus -\phi(f) \cong \phi([f,g]) \cong \phi(e) \cong \mathbf{0}$$

of the neutral element $\mathbf{0} \in BA$, which corresponds to the automorphism

$$\phi(fg) \cong \phi(f)\phi(g) \cong \phi(g)\phi(f) \cong \phi(gf) \cong \phi(fg)$$

of $\phi(fg)$.

3.2.5. *A More Precise Picture.* Since Σ^∞ is a left adjoint it preserves colimits. We obtain therefore a pushout diagram

$$\begin{tikzcd}
\Sigma^\infty(\mathbb{S}^1) \arrow[r] \arrow[d, "{[f,g]}"'] & \bullet \arrow[d] \\
\Sigma^\infty(\mathbb{S}^1 \vee \mathbb{S}^1) \arrow[r] & \Sigma^\infty(\mathbb{T}^2)
\end{tikzcd}$$

in the category of spectra Sp by applying the functor Σ^∞ to diagram (9).

Remark 3.24. *The map $[f,g]\colon \Sigma^\infty\mathbb{S}^1 \longrightarrow \Sigma^\infty(\mathbb{S}^1 \vee \mathbb{S}^1)$ is homotopic to zero, since it is defined as the adjoint of a commutator in the E_∞-object $\Omega^\infty\Sigma^\infty(\mathbb{S}^1 \vee \mathbb{S}^1)$.*

In particular, we obtain that the map $\bullet \longrightarrow \mathbb{S}^1 \vee \mathbb{S}^1$ induces a map from the pushout diagram

(10)
$$\begin{tikzcd}
\Sigma^\infty(\mathbb{S}^1) \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] & \Sigma^\infty(\mathbb{S}^2).
\end{tikzcd}$$

This gives rise to a splitting of the distinguished square

$$\begin{tikzcd}
\Sigma^\infty\mathbb{S}^1 \oplus \Sigma^\infty\mathbb{S}^1 \arrow[r] \arrow[d] & \Sigma^\infty\mathbb{T}^2 \arrow[d] \\
\bullet \arrow[r] & \Sigma^\infty\mathbb{S}^2, \arrow[u, dotted, bend right]
\end{tikzcd}$$

thereby reproving the second assertion of Lemma 3.11, and showing the compatibility of Definition 3.12 of higher commutators and the picture developed in paragraph 3.2.4:

By virtue of this splitting, we associate to any map $\Sigma^\infty\mathbb{T}^2 \longrightarrow \Sigma^2\mathbb{E}$, a morphism $\Sigma^\infty\mathbb{S}^2 \longrightarrow \Sigma^2\mathbb{E}$, which in turn represents the higher commutator in $\pi_0(\mathbb{E})$. Consequently, we have the following lemma, which is a generalization of Remark 3.23.

Lemma 3.25. *Let $e\colon BG \longrightarrow \Omega^\infty\Sigma^2\mathbb{E}$ be the infinite looping of a spectral extension of G by $\mathbb{E}$. The datum of such a map is equivalent to an E_1-map $\phi\colon G \longrightarrow \Omega^\infty\Sigma\mathbb{E}$ Given $(f,g) \in P_2(G)$, the infinite looping of the higher commutator $f \star g\colon \mathbb{S}^2 \longrightarrow \Omega^\infty\Sigma\mathbb{E}$ corresponds to the loop in $\Omega^\infty\Sigma\mathbb{E}$ induced by*

$$\phi(fg) \cong \phi(f)\phi(g) \cong \phi(g)\phi(f) \cong \phi(gf) \cong \phi(fg).$$

Proof. A map $\Sigma^\infty\mathbb{S}^2 \longrightarrow \Sigma^2\mathbb{E}$ corresponds to a map $\Sigma^\infty\mathbb{S}^1 \longrightarrow \Sigma\mathbb{E}$. This map was described as the descent data in paragraph 3.2.4, and is obtained as the difference of the trivialization of Remark 3.24, which stated $\phi([f,g]) \cong \mathbf{0}$ using the E_∞-structure of $\mathbb{E}$, and the trivialization $\phi(f)\phi(g) \cong \phi(g)\phi(f)$, induced by $fg = gf$. $\square$

3.3. **Examples of Higher Commutators.**

3.3.1. *Graded Extensions.* A *graded central extension* of G by an abelian group A is a pair (e, ν), where e is a central extension as in equation (7), and ν is a group homomorphism $G \longrightarrow \mathbb{Z}$. The graded commutator of $(f, g) \in P_2(G)$ is by definition $(-1)^{\nu(f)\nu(g)} f \star g$.

Definition 3.26. *We denote by $B^{\mathbb{Z}}A$ the groupoid of* graded A-torsors, *i.e. pairs (M, n), where M is an A-torsor and n an integer. One has a structure of a Picard groupoid on $B^{\mathbb{Z}}A$, given by $\otimes_A$ of A-torsors, and addition of integers. The symmetry constraint for $(M, m), (N, n)$ is induced by the one of $\otimes_A$, twisted by the sign $(-1)^{mn}$. We denote this Picard groupoid by $\mathbb{B}^{\mathbb{Z}}A$.*

Using $\mathbb{B}^{\mathbb{Z}}A$ we can give a more conceptual treatment of graded central extensions, we record the statement and leave the verification to the reader.

Lemma 3.27. *The groupoid of graded central extensions of G by A is equivalent to the groupoid of monoidal maps $\phi\colon G \longrightarrow \mathbb{B}^{\mathbb{Z}}A$. The graded commutator*

$$(-1)^{\nu(f)\nu(g)} f \star g$$

agrees with the automorphism of $\phi(fg)$ defined by

$$\phi(fg) \cong \phi(f)\phi(g) \cong \phi(g)\phi(f) \cong \phi(gf) \cong \phi(fg).$$

A monoidal morphism $G \longrightarrow B^{\mathbb{Z}}A$ map corresponds to a map of spaces

$$BG \longrightarrow BB^{\mathbb{Z}}A. \tag{11}$$

Viewing the Picard groupoid $\mathbb{B}^{\mathbb{Z}}A$ as an object in $\mathsf{Sp}_{[0,1]}$, the adjoint of (11) defines a *spectral extension* of G by the non-connective spectrum $\Omega\mathbb{B}^{\mathbb{Z}}A$.

We can now prove the following comparison between commutators for spectral extensions and graded commutators. As a special case of Lemma 3.25 we obtain the following result.

Corollary 3.28. *Let $e\colon BG \longrightarrow B(B^{\mathbb{Z}}A)$ be the map corresponding to a graded central extension of G by A. For every $(f, g) \in P_2(G)$ one has that $f \star_e g$ agrees with the graded commutator.*

3.3.2. *Steinberg Symbols.* In the following we denote by R a ring, which is not necessarily assumed to be commutative. Careful inspection of the definition of your choice of algebraic K-theory, reveals the existence of a canonical morphism

$$\coprod_{n \in \mathbb{N}} \Sigma^\infty B\,\mathrm{GL}_n(R) \longrightarrow \mathbb{K}_R\,. \tag{12}$$

More generally, for a stable ∞-category C, there is a canonical morphism

$$\Sigma^\infty \mathsf{C}^\times \longrightarrow \mathbb{K}_{\mathsf{C}}\,. \tag{13}$$

The morphism (12) is a special case of this construction.[3]

Definition 3.29. *The existence of the morphism* (12) *can be restated as saying that $\coprod_{n\in\mathbb{N}} B\,\mathrm{GL}_n(R)$ is canonically endowed with a central extension by $\Omega^2\,\mathbb{K}_R$. Similarly,* (13) *can be restated as saying that $\mathsf{C}^\times$ is canonically endowed with a central extension by $\Omega^2\,\mathbb{K}_{\mathsf{C}}$. We will denote the extensions by e_R and e_{C} respectively.*

The central extension of $\mathrm{GL}_n(R)$ by $\Omega^2\,\mathbb{K}_R$ has appeared in work of Safronov [Saf13]. The theory of higher commutators developed in 3.2.3 enables us to generalize *Steinberg symbols* to a general stable ∞-category.

Definition 3.30. *In the notation of Definitions 3.18 and 3.19, let $(g_1, \ldots, g_n) \in P_n(\mathsf{C}^\times, x)$ be an n-tuple of commuting automorphisms in a stable ∞-category C represented by a map $\mathbb{T}^n \longrightarrow \mathsf{C}^\times$. We define*

$$\{g_1, \ldots, g_n\}\colon \Sigma^\infty \mathbb{S}^n_+ \longrightarrow \mathbb{K}_{\mathsf{C}}$$

as the higher commutator with respect to the natural extension of $\mathsf{C}^\times$ by $\Omega^2\,\mathbb{K}_{\mathsf{C}}$.

[3] i.e. after factoring through the inclusion $P_f(R)^\times \longrightarrow \mathrm{Perf}(R)^\times$.

The justification of the terminology *Steinberg symbol* is provided by the next proposition, which compares the higher commutators of Definition 3.30 with Loday's higher Steinberg symbols, for the category of finitely generated projective R-modules.

Proposition 3.31. *Let R be a commutative ring, and $r_1, \dots, r_n \in R^\times$ be an n-tuple of units in R. The higher commutator $r_1 \star \cdots \star r_n$, computed with respect to the spectral extension e_R of Definition 3.29, agrees with Loday's higher Steinberg symbol $\{r_1, \dots, r_n\}$.*

Before giving the proof, we recall Loday's definition from [Lod76]. In modern language, Loday's construction of the Steinberg symbols relies on the E_∞-ring structure of K_R (in which the product is induced by the tensor product $\otimes$ of R-modules). If $\alpha_1, \dots, \alpha_n$ is an n-tuple of paths in K_R based at $\mathbf{0} \in K_R$, the multiplication $\otimes$ induces a map

$$\Sigma^\infty \mathbb{S}^{\nVdash \wedge n} \longrightarrow \mathbb{K}_R^{\wedge n} \longrightarrow \mathbb{K}_R,$$

which defines an element $\alpha_1 \cup \cdots \cup \alpha_n$ of $\pi_n(K_R) = K_n(R)$.

The n-tuple $(r_1, \dots, r_n)$ of commuting units gives rise to an n-tuple of paths $(\tilde{\alpha}_1, \dots, \tilde{\alpha}_n)$ in K_R by means of the natural map (12). However, the paths are based at $\mathbf{1} \in K_R$. Loday rectifies this by considering instead the paths $\alpha_i = \tilde{\alpha}_i \ominus \mathbf{1}$.

Proof of Proposition 3.31. We will draw two diagrams and verify their homotopy commutativity. The asserted identity will then be a consequence of these considerations. We emphasize that we have omitted the functor Σ^∞ for the sake for readability.

$$\begin{tikzcd}
(B\,\mathrm{GL}_1(R)^{\times n})_+ \arrow[r] \arrow[d] & (B\,\mathrm{GL}_1(R^{\otimes n}))_+ \arrow[r] \arrow[d] & (B\,\mathrm{GL}_1(R))_+ \arrow[d] \\
\mathcal{K}_R^{\wedge n} \arrow[r] & \mathcal{K}_{R^{\otimes n}} \arrow[r] & \mathcal{K}_R
\end{tikzcd}$$

The vertical arrows are induced by the canonical map $B\,\mathrm{GL}_1(R)_+ \longrightarrow \mathcal{K}_R$, which sends the adjoined base point to zero and $B\,\mathrm{GL}_1(R)$ to the connected component of $\mathcal{K}_R$ corresponding to $\mathbf{1} \in K_0(R)$. The left horizontal map at the bottom of the diagram is the smash product of the maps induced by tensoring along the n inclusion-of-a-factor maps $R \longrightarrow R^{\otimes n}$. The right horizontal map is induced by tensor product along the multiplication map $R^{\otimes n} \longrightarrow R$.

Omitting the inner column, and accounting for the maps $\mathbb{S}^1 \longrightarrow B\,\mathrm{GL}_1(R)$ representing the elements $r_1, \dots, r_n$ we obtain the following commutative diagram:

$$\begin{tikzcd}
 & ((\mathbb{S}^1)^{\times n})_+ \arrow[dl] \arrow[d] \\
(B\,\mathrm{GL}_1(R)^{\times n})_+ \arrow[r] \arrow[d] & B\,\mathrm{GL}_1(R)_+ \arrow[d] \\
\mathcal{K}_R^{\wedge n} \arrow[r] & \mathcal{K}_R\,.
\end{tikzcd}$$

Replacing the loops represented by $(r_1, \dots, r_n)$ by the shifted loops given by $r_i \ominus \mathbf{1}$, we see that the pushforward of the class in $\pi_n(\Omega^\infty \mathcal{K}_R, \mathbf{1}) \cong \pi_n(\mathcal{K}_R)$ agrees with the Steinberg symbol $\{r_1, \dots, r_n\}$. Here we use the fact that the isomorphism

$$\pi_n(\Omega^\infty \mathcal{K}_R, \mathbf{1}) \cong \pi_n(\mathcal{K}_R)$$

is induced by the map $- \ominus \mathbf{1} \colon (\Omega^\infty \mathcal{K}_R, \mathbf{1}) \longrightarrow (\Omega^\infty \mathcal{K}_R, 0)$. We conclude that the higher commutator $r_1 \star \cdots \star r_n$ is indeed the Steinberg symbol $\{r_1, \dots, r_n\}$. $\square$

3.4. **Recursion.** The goal of this subsection is to compare our commutator construction with the one introduced by Osipov–Zhu ([OZ11]) in the context of central extensions of groups by Picard groupoids. In order to accomplish this, we develop a recursive algorithm to compute a higher commutator $(g_1 \star_e \cdots \star_e g_n)$. The recursion runs over n. To an element $g \in G$ we will associate a spectral extension $e\langle g\rangle$ of the centralizer $C(g)$, such that

$$g_1 \star_e \cdots \star_e g_n = g_2 \star_{e\langle g_1\rangle} \cdots \star_{e\langle g_1\rangle} g_n.$$

This corresponds, in group cohomology, to the slant-product with the cycle g.

3.4.1. *An Inductive Approach to Higher Commutators.* Let $g \in G$ be an element of the group G, and $e\colon \Sigma^\infty BG \longrightarrow \Sigma^2\mathbb{E}$ a spectral extension. We will refer to the centralizer of g in G by the notation $C(g)$. The element g induces a group homomorphism $\mathbb{Z} \times C(g) \longrightarrow G$, sending (n, h) to $g^n h$. Applying the classifying space functor, we obtain a map of spaces

$$\mathbb{S}^1 \times BC(g) \longrightarrow BG.$$

Using the stable splitting of products (Lemma 3.11) we obtain a natural morphism

$$\Sigma\Sigma^\infty BC(g) \cong \Sigma^\infty\mathbb{S}^1 \wedge \Sigma^\infty BC(g) \longrightarrow \Sigma^2\mathbb{E},$$

which corresponds to a spectral extension

$$\Sigma^\infty BC(g) \longrightarrow \Sigma^2\Omega\mathbb{E} \tag{14}$$

of $C(g)$ by $\Omega\mathbb{E}$.

Definition 3.32. *The spectral extension of $C(g)$ by $\Omega\mathbb{E}$ will be denoted by $e\langle g\rangle$.*

Using the shift-construction $e\langle g\rangle$, we can compute higher commutators inductively by virtue of the following result.

Lemma 3.33. $(g_1\sharp \dots \sharp g_n)_e \cong (g_2\sharp \dots \sharp g_n)_{e\langle g_1\rangle}$.

Proof. The right hand side of the equation above is represented by the composition

$$\Sigma^\infty\mathbb{S}^{n-1} \longrightarrow \Sigma^\infty\mathbb{T}^{n-1} \xrightarrow{g_2,\dots,g_n} \Sigma^\infty BC(g_1) \longrightarrow \Sigma^2\Omega\mathbb{E} \cong \Omega\Sigma^2\mathbb{E}.$$

This map on the other hand is adjoint to

$$\Sigma^\infty\mathbb{S}^{n} \longrightarrow \Sigma^\infty\mathbb{T}^{n} \xrightarrow{g_1,\dots,g_n} \Sigma^\infty(\mathbb{S}^1 \times BC(g_1)) \longrightarrow \Sigma^2\mathbb{E},$$

which represents the left hand side of the identity above. □

3.4.2. *Comparison with Osipov–Zhu's $C_3(f, g, h)$.* A sceptic might assert that the definition of higher commutators is too abstract in order to be amenable to direct computation. We claim that this is not the case, although it might be difficult to evaluate $g_1\sharp \dots \sharp g_n$ directly for $n \geq 3$. If $n = 2$ however, determining $g_1\sharp g_2$ is in general not much different from a graded commutator (see Corollary 3.28). We explain in this paragraph how one can reduce always to this case.

Given a spectral extension e of G by $\mathbb{E}$ and an element $g \in G$, we have constructed a spectral extension $e\langle g\rangle$ of the centralizer $C(g)$ by $\Omega\mathbb{E}$. This shifted spectral extension satisfies the identity (Lemma 3.33)

$$(g_1 \star \cdots \star g_n)_e \cong (g_2 \star \cdots \star g_n)_{e\langle g_1\rangle}.$$

Readers of [OZ11] will recognize the similarity with the recursive computation in [OZ11]. The authors of *loc. cit.* associate to an extension of a group G by a Picard groupoid $\mathbf{P}$, and an element $f \in G$ a (graded) central extension of $C(f)$ by $\Omega\mathbf{P} \cong \mathrm{Aut}_{\mathbf{P}}(\mathbf{0})$. Eventually, the commutator $C_3(f, g, h)$ is defined to be $C_2(g, h)$ with respect to the latter central extension.

Proposition 3.34. *Let $\phi\colon G \longrightarrow B\mathbf{P}$ be the monoidal map corresponding to a central extension of G by $\mathbf{P}$. We denote by $e\colon \Sigma^\infty BG \longrightarrow \Sigma\mathbb{B}\mathbf{P}$ the corresponding spectral extension of G by $\Omega\mathbb{B}\mathbf{P}$, the spectrum associated to the Picard groupoid $\mathbf{P}$. Then,*

$$C_3(f, g, h) = f \star g \star h.$$

Proof. In light of Lemma 3.25 it suffices to understand

$$e\langle f\rangle\colon \Sigma^\infty BC(f) \longrightarrow \mathbb{B}\mathbf{P}. \tag{15}$$

Since $\pi_0 BC(f) = 0$, $e\langle f\rangle$ factors through the *connected cover*

$$\tau_{\geq 1}\mathbb{B}\mathbf{P} \cong \mathbb{B}B\Omega\mathbf{P}.$$

The discussion in paragraphs 3.2.4 and 3.2.5 reveals that this factorization corresponds to the monoidal map

$$C(f) \longrightarrow \Omega\mathbf{P},$$

which sends $g \in C(f)$ to the element of $Aut_{\mathbf{P}}(\mathbf{0})$ given by

$$\phi(fg) \cong \phi(f)\phi(g) \cong \phi(g)\phi(f) \cong \phi(gf) \cong \phi(fg).$$

This is precisely how Osipov–Zhu define the central extension of $C(f)$ by $\Omega\mathbf{P} = Aut_{\mathbf{P}}(\mathbf{0})$ in Lemma-Definition 2.5 of [OZ11]. □

4. Symbols

This section introduces the *higher-dimensional Contou-Carrère symbol* $(f_0, \ldots, f_n) \in A^\times$ for elements $f_0, \ldots, f_n \in A((t_1))\cdots((t_n))^\times$. We will compare it to its historical predecessor, the *tame symbol*, and outline the similarities and differences between the two types of symbols.

4.1. **The Tame Symbol.**

4.1.1. *The One-Dimensional Tame Symbol.* Let R be a discrete valuation ring with valuation ν. We denote its residue field by κ and its fraction field by F. The tame symbol is defined to be

$$(f, g) = (-1)^{\nu(f)\nu(g)}\overline{\left(\frac{f^{\nu(g)}}{g^{\nu(f)}}\right)} \in \kappa^\times, \tag{16}$$

where the overline refers to sending the unit to a unit in the residue field. The following result is well-known, appearing, for instance, in Milnor's [Mil70, p. 323].

Proposition 4.1. *Let $\partial\colon K_2(F) \longrightarrow K_1(\kappa)$ denote the boundary morphism of Quillen's localization sequence. Under the natural identification $K_1(\kappa) = \kappa^\times$, we have*

$$(f, g) = \partial(\{f, g\}),$$

where $\{-,-\}$ refers to the image of $f, g \in K_1(F)$ under the product $K_1(F) \times K_1(F) \longrightarrow K_2(F)$ (i.e. to the Steinberg symbol).

The aforementioned result is only one instance of how the tame symbol (16) is connected to geometry. In [Del91] Deligne studied the tame symbol in the complex-analytic category, in which he described (f, g) as the monodromy of a natural flat (holomorphic) connection on a punctured disc associated to the pair f and g. While this picture is not applicable in the algebraic framework we are concerned with, it justifies thinking of (16) as a measure of monodromy.

4.1.2. *Relation to Number Theory.* The name "tame symbol" for the map (16) originates from number theory: Let F be an ordinary local field with finite residue field (e.g. $\mathbb{F}_p((t))$ or the p-adics $\mathbb{Q}_p$), $\kappa := \mathbb{F}_q$ its residue field and $p = \operatorname{char}\kappa$ its characteristic. Let $\mu(F) = F^\times_{tor}$ be the group of roots of unity, $\hat{m}$ its cardinality and $\hat{m} = p^r(q-1)$ its factorization in the p-part and prime-to-p part, reflecting the decomposition

$$\mu(F) = \mu(F)\{p\text{-primary part}\} \oplus \mathbb{F}_q^\times; \tag{17}$$

the roots of unity of prime-to-p order are in bijection with the roots of unity of the residue field. There is a canonical isomorphism

$$\operatorname{hilb}_F : K_2(F)/\hat{m}K_2(F) \longrightarrow \mu(F), \tag{18}$$

known as the *Hilbert symbol*. It is related to arithmetic questions, namely

$$\{\alpha,\beta\}\mapsto 1 \quad\Leftrightarrow\quad \begin{array}{l}\beta \text{ can be written as a norm of an element}\\ \text{in the extension field } F[\sqrt[n]{\alpha}],\end{array}$$

and determines the essential part of $K_2(F)$; Merkurjev proved that $K_2(F)\simeq \mu(F)\oplus V$, where V is a $\mathbb{Q}$-vector space of uncountable dimension [Mer83]. Splitting the isomorphism of (18) into the p-part and prime-to-p part as in (17), one finds precisely the *same* simple explicit formula as in (16) for the prime-to-p part. Attempting to use the same formula for the p-part yields the zero map − obviously so, as the multiplicative group of the residue field is p-torsion free. Indeed explicit formulae for the p-part are much more complicated and have a rich theory [FV02, Ch. 7-9];

4.1.3. *The Higher-Dimensional Tame Symbol.* In order to prepare the stage for the generalization of the tame symbol to higher dimensions, we need to introduce a higher-dimensional analogue of discretely valued fields. We refer the reader to [Mor] for an expository account.

Definition 4.2. *We denote by F a field. The* structure of a 1-dimensional field *on F is given by an isomorphism $F=\mathrm{Frac}\,R$, where R is a discrete valuation ring. Inductively, we say that the* structure of an n-dimensional field *on F is an isomorphism $F=\mathrm{Frac}\,R$, where R is a discrete valuation ring, and the residue field $F^{(n-1)}$ of R is endowed with the structure of an $(n-1)$-dimensional field.*

It is important to think of an n-dimensional field as a field endowed with an extra structure. Moreover, the Krull dimension of a field is 0, hence one needs to be careful when speaking of the *dimension of a field*. We therefore use the convention in the present paper that the Krull dimension of a ring will always be explicitly referred to as the Krull dimension. If we speak of the dimension of a scheme though, we will refer to the dimension in the usual scheme-theoretic sense, which coincides with the Krull dimension in the affine case.

Remark 4.3. *If $F^{(n)}$ is an n-dimensional field so that all $F^{(i)}$ are complete discrete valuation fields, then $F^{(n)}$ is (non-canonically) isomorphic to $F^{(0)}((t_1))\dots((t_n))$. This is a corollary of a theorem of Cohen* [Coh46].

Let $F^{(n)}$ be an n-dimensional field. Denote by $R^{(n)}$ the discrete valuation ring corresponding to $F^{(n)}$, i.e. the subring of $F^{(n)}$ given by elements x with $\nu(x)\geq 0$. Its residue field will be referred to by $F^{(n-1)}$. The resulting setup can be depicted as a zigzag diagram

$$\begin{tikzcd}
\kappa = F^{(0)} \arrow[d, dash] & & F^{(1)} \arrow[ll, dotted] \arrow[dl, dash] & \cdots & F^{(n-1)} \arrow[d, dash] & & F^{(n)}, \arrow[ll, dotted] \arrow[dl, dash] \\
 & R^{(1)} & & & & R^{(n)} &
\end{tikzcd}$$

where the pointed arrows are understood to represent the boundary maps in the localization sequence

$$\cdots\longrightarrow K_i(F^{(k-1)})\longrightarrow K_i(R^{(k)})\longrightarrow K_i(F^{(k)})\xrightarrow{\partial^{(k)}} K_{i-1}(F^{(k-1)})\longrightarrow\cdots.$$

Proposition 4.1 described the tame symbol as a boundary map in K-theory. This motivates the following definition of the higher-dimensional tame symbol, which was introduced by Parshin in [Par76], for $n=2$, and by Kato in [Kat86], for $n>2$.

Definition 4.4. *Let $F^{(n)}$ be an n-dimensional field. The* higher tame symbol *is defined to be the composition*

$$\partial^{(1)}\circ\cdots\circ\partial^{(n)}\colon K^M_{n+1}(F^{(n)})\longrightarrow K_1(\kappa)=\kappa^\times.$$

Zigzag diagrams as the one above, and in particular the resulting composition of boundary morphisms, will make a reappearance in 4.3.1.

If $f_0,\dots,f_{n-1}\in(R^{(n)})^\times$, and π denotes a generator of the maximal ideal of the discrete valuation ring $R^{(n)}$, then one has

$$(f_0,\dots,f_{n-1},\pi)=(\bar f_0,\dots,\bar f_{n-1}),$$

where $\bar{f}_i \in F^{(n-1)}$ denotes the associated class in the residue field (see [Mil70, Lemma 2.1]). Using this identity and the formal properties of the Steinberg symbol (multilinearity, alternating, Steinberg relation), higher tame symbols can be evaluated recursively.

In the complex-analytic theory, the tame symbol $(f_0, \ldots, f_n)$ was understood as higher monodromy of higher gerbes associated to the $(n+1)$-tuple of functions, by Brylinski and McLaughlin in [BM96a] and [BM96b], thereby extending the aforementioned work of Deligne [Del91].

4.2. **Higher-Dimensional Contou-Carrère Symbols.**

4.2.1. *Loop Groups and Spectral Extensions.* Recall that the loop group $L\,\mathbb{G}_m$ is defined as the group-valued sheaf

$$L\,\mathbb{G}_m\colon (\mathrm{Aff}_k)^{\mathrm{op}} \longrightarrow \mathsf{Grp}$$

sending $\operatorname{Spec} A$ to $A((t))^\times$. We denote by $\mathcal{V}_A$ the Tate object $A((t))$ in $\mathsf{Tate}(P_f(A))$. There is a natural map

$$L\,\mathbb{G}_m(A) \longrightarrow \mathrm{Aut}(\mathcal{V}_A)$$

for every k-algebra A. For each A, the index map and determinant give rise to a spectral extension

$$\Sigma^\infty BL\,\mathbb{G}_m(A) \longrightarrow \Sigma^\infty B\,\mathrm{Aut}(\mathcal{V}_A) \longrightarrow \mathcal{K}_{\mathsf{Tate}^{el}(P_f(A))} \xrightarrow{\mathsf{Index}} \Sigma\,\mathcal{K}_{P_f(A)} \xrightarrow{\det} \Sigma\mathbb{B}^{\mathbb{Z}}A^\times.$$

Looping the adjoint of this map yields an E_1-map

$$L\,\mathbb{G}_m(A) \longrightarrow B^{\mathbb{Z}}\,\mathbb{G}_m(A),$$

classifying a graded central extension of $L\,\mathbb{G}_m(A)$. The construction is natural in maps $A \longrightarrow A'$, so it defines a central extension of group-valued sheaves. We record this observation in the following definition.

Definition 4.5. *The graded central extension of $L\,\mathbb{G}_m$ will be denoted by*

$$\phi_{KM}\colon L\,\mathbb{G}_m \longrightarrow B^{\mathbb{Z}}\,\mathbb{G}_m,$$

and referred to as the Kac–Moody extension *of the loop group. It is induced by truncating the so-called* spectral Kac–Moody extension*, which we denote by*

$$e_{sKM}\colon \Sigma^\infty BL\,\mathbb{G}_m \longrightarrow \mathcal{K},$$

where $\mathcal{K}$ denotes the presheaf in connective spectra, sending a ring A to $\mathcal{K}_A$.

We can now recall the following well-known result, which generalizes the main result of the paper [APR04] to arbitrary k-algebras (without restricting to the artinian case).

Proposition 4.6. *The graded central extension ϕ_{KM} of Definition 4.5 relates to the Contou-Carrère symbol by means of the relation*

$$(-,-)^{-1} = - \star_{\phi_{KM}} -.$$

Proof. Proposition 3.3 of [BBE02] verifies that the classical notion of the Kac–Moody extension of loop groups has this property. In [BGW14a, Par. 5.2.3] we compare the the extension ϕ_{KM} with its classical definition in terms of determinant lines. □

Definition 4.7. *The n-fold loop group L^nG of a group-valued sheaf G is defined to be the group-valued sheaf which sends the affine scheme* $\operatorname{Spec} A$ *to $G(A((t_1))\ldots((t_n)))$.*

There is an analogue of the Kac–Moody extension for loop groups. Denoting by $\mathcal{V}_A$ the n-Tate object $A((t_1))\ldots((t_n))$ in $\mathsf{n}\text{-}\mathsf{Tate}(P_f(A))$, we have a natural map

$$L^n\,\mathbb{G}_m(A) \longrightarrow \mathrm{Aut}(\mathcal{V}_A)$$

for every k-algebra A. The index map gives rise to a spectral extension

$$\Sigma^\infty BL^n\,\mathbb{G}_m(A) \longrightarrow \Sigma^\infty B\,\mathrm{Aut}(\mathcal{V}_A) \longrightarrow \mathbb{K}_{\mathsf{n}\text{-}\mathsf{Tate}(P_f(A))} \xrightarrow{\mathsf{Index}^n} \Sigma^n\,\mathbb{K}_{P_f(A)}$$

of $L^n\,\mathbb{G}_m(A)$ by $\Sigma^{n-2}\,\mathbb{K}_{P_f(A)}$. As above, the construction is natural in maps $A \longrightarrow A'$, so it defines a central extension of sheaves in groups. We record this observation in the following definition.

Definition 4.8. *The spectral extension of $L^n \mathbb{G}_m$ by $\Sigma^{n-2} \mathbb{K}$ constructed above will be referred to as the canonical spectral extension of the n-fold loop group $L^n \mathbb{G}_m$. We denote the corresponding map of spectra by e_n.*

As an application of this construction we give a definition of higher Contou-Carrère symbols.

Definition 4.9. *Let $f_0, \dots, f_n \in L^n \mathbb{G}_m(A) = A((t_1)) \dots ((t_n))^\times$. We denote by* det *the determinant map $K_1(A) \longrightarrow A^\times$. The Contou-Carrère symbol $(f_0, \dots, f_n)$ is defined to be the higher commutator*

$$\det((f_0 \star \dots \star f_n)_{e_n}^{(-1)^n}).$$

The study of the higher-dimensional Contou-Carrère symbol $(f_0, \dots, f_n)$ for an $(n+1)$-tuple in $A((t_1)) \dots ((t_n))$, with A a k-algebra, has been pioneered by Osipov–Zhu in the case of $n = 2$ (see [OZ13]). They identified this symbol with a higher commutator in a central extension of the double loop group $L^2 \mathbb{G}_m$ by $B^{\mathbb{Z}} \mathbb{G}_m$. Inspired by this observation and the one-dimensional case (Proposition 4.6), they define the two-dimensional Contou-Carrère symbol for general k-algebras A as a higher commutator $C_3(f, g, h)$.

Proposition 4.10. *Definition 4.9 is compatible with the definition of Contou-Carrère in dimension* 1*, and Osipov–Zhu in dimension* 2*.*

The proof of the 1-dimensional case was the content of Proposition 4.6. We now turn to verifying the assertion for $n = 2$.

Proof of the 2*-dimensional case:* Osipov–Zhu construct a central extension of $L^2 \mathbb{G}_m$ by the Picard groupoid $B^{\mathbb{Z}} \mathbb{G}_m$ ([OZ13, p. 28]), and define (f, g, h) for a triple in $A((t_1))((t_2))^\times$, as the higher commutator $C_3(f, g, h)$. We have seen in Proposition 3.34 that $f \star g \star h = C_3(f, g, h)$. So to conclude the assertion, we need to verify that for $n = 2$ the spectral extension of $L^n \mathbb{G}_m$ constructed in Definition 4.8 is related to the extension

$$\Sigma^\infty BL^2 \mathbb{G}_m \longrightarrow \Sigma^2 \mathbb{B}^{\mathbb{Z}} \mathbb{G}_m .$$

constructed by Osipov–Zhu. In [BGW14a, Section 5.2] we explained how Nisnevich-local vanishing of K_{-1} (see [Dri06, Thm. 3.7]) can be used to construct a morphism $\Sigma^\infty BL^2 \mathbb{G}_m(A) \longrightarrow \Sigma^2 \mathbb{B}^{\mathbb{Z}} \mathbb{G}_m$. In more detail, by Nisnevich descent, it suffices to consider rings A with $K_{-1}(A) = 0$. We then have a commutative diagram

$$\begin{tikzcd}
\Sigma^\infty BL^2 \mathbb{G}_m(A) \arrow[r] \arrow[dr, "e_2"'] & \Sigma^2 \mathcal{K}_A \arrow[r, "\det^{\mathbb{Z}}"] \arrow[d] & \Sigma^2 \mathbb{B}^{\mathbb{Z}} \mathbb{G}_m(A) \\
 & \Sigma^2 \mathbb{K}_A . &
\end{tikzcd}$$

Using the adjunction between Σ^∞ and Ω^∞, we obtain a map

$$e_2 \colon BL^2 \mathbb{G}_m \longrightarrow B^2 B^{\mathbb{Z}} \mathbb{G}_m .$$

Looping once yields an E_1-map to the classifying space of the Picard groupoid of graded lines $B^{\mathbb{Z}} \mathbb{G}_m$

$$\phi \colon L^2 \mathbb{G}_m \longrightarrow BB^{\mathbb{Z}} \mathbb{G}_m .$$

We have to show that this morphism is -1 times of the one constructed by Osipov–Zhu.

According to [BGW14a, p. 35 & Cor. 4.28], ϕ sends $f \in L^2 \mathbb{G}_m(A)$ to

$$\det{}^{\mathbb{Z}}(N/\phi L) \otimes \det{}^{\mathbb{Z}}(N/L)^\vee,$$

for N a lattice containing both ϕL and L, with the monoidal structure being defined in terms of common enveloping lattices. This is precisely the dual of the definition given by Osipov–Zhu [OZ13, p. 28]. □

A study of Contou-Carrère symbols for surfaces, using analytic techniques has appeared in work of Horozov–Luo [HL13].

4.2.2. *A Reminder on Beilinson-Parshin Adèles.* For a curve X, one defines the adèles $\mathbb{A}_X$ as the *restricted product*

$$\prod_{x\in X_0}{}' \operatorname{Frac}\widehat{\mathcal{O}_{X,x}},$$

where the product ranges over closed points of X, and we stipulate that all but finitely many factors lie in $\widehat{\mathcal{O}_{X,x}}$. The correct generalization of this concept to higher-dimensional varieties requires more terminology.

From now on, X denotes an excellent scheme. The set of points underlying the scheme X will be denoted by $|X|$. It is naturally endowed with a partial ordering $\prec$, where we say that $x \prec y$ if x lies in $\overline{\{y\}}$.

Definition 4.11. *Let $S := (S, \prec)$ be a partially ordered set. We denote by $S_\bullet$ the associated simplicial set, where S_k is the set of chains $x_0 \prec \cdots \prec x_k$ in S of length $k+1$, with the obvious boundary and degeneration maps.*

In particular we obtain a simplicial set $|X|_\bullet$ for every scheme X. A k-simplex, i.e. an element $\xi \in |X|_k$, corresponds to a length $k+1$ flag of reduced, irreducible subschemes $Z_0 \subset \cdots \subset Z_k$ of X. A *reduced* simplex is given by a flag with $Z_i \neq Z_{i+1}$ for all i.

Definition 4.12. *We denote by $|X|_\bullet^{\leq d}$ the simplicial subset, which consists of all simplices $x_0 \prec \cdots \prec x_k$, such that $\dim \overline{\{x_k\}} \leq d$. We refer to the set of reduced k-simplices in $|X|_\bullet$ by $|X|_n^{\mathrm{red}}$.*

We will also need the following definition to introduce Beilinson adèles.

Definition 4.13. *For $T \subset |X|_{k+1}^{\mathrm{red}}$ and $x \in |X|$ we let ${}_xT \subset |X|_k^{\mathrm{red}}$ denote the reduced simplicial subset consisting of k-simplices ξ of $|X|_\bullet$, such that the concatenation $x_0 \prec \cdots \prec x_k \prec x$ is well-defined and lies in T.*

If X is an irreducible scheme of dimension n, and η is the generic point, then ${}_\eta(|X|_\bullet^{\leq n}) = |X|_\bullet^{\leq n-1}$.

Now we are ready to give Beilinson's definition ([Bei80]), in the variant of *reduced adèles* as defined by Huber in [Hub91].

Definition 4.14 (Beilinson)**.** *Let X be a Noetherian scheme, and for $n \in \mathbb{N}$ we let $T \subset |X|_n^{\mathrm{red}}$. There exists a unique family of exact functors*

$$\mathbb{A}(T,-) : \mathrm{QCoh}(X) \longrightarrow \mathsf{Mod}(\mathcal{O}_X),$$

indexed by such sets T, satisfying the following relations.

(a) *If $n = 0$, then $\mathbb{A}_X(T, \mathcal{F}) = \prod_{x\in T} \widehat{\mathcal{F}}_x$.*
(b) *We denote by $j_{rx}\colon \operatorname{Spec}\mathcal{O}_{X,x}/\mathfrak{m}_x^r \longrightarrow X$ the canonical map. Let $\mathbf{j}_{rx} = (j_{rx})_* j_{rx}^*$ be the functor from the category $\mathsf{Mod}(\mathcal{O}_X)$ to itself. For $\mathcal{F} \in \mathrm{Coh}(X)$ we have $\mathbb{A}_X(T, \mathcal{F}) = \prod_{x\in X} \varprojlim_{r\geq 0}(\mathbb{A}_X({}_xT, j_{rx}\,\mathcal{F}))$, where all limits are taken in the category $\mathsf{Mod}(\mathcal{O}_X)$.*
(b) *The functors $\mathbb{A}_X(T,-)$ commute with directed colimits.*

For X a curve, and $T = |X|_\bullet$ we recover the classical adèles by taking global sections

$$H^0(X, \mathbb{A}(T, \mathcal{O}_X)) = \mathbb{A}_X\,.$$

The adèles of a surface were introduced by Parshin in [Par76]. In [Bei80], Beilinson extended this definition to schemes of arbitrary dimension. The general definition of adèles, albeit important to have, is sometimes hard to decipher computationally. In the local case, relevant for us, there exists an alternative approach, via iterated completion and localization at a chain of ideals. A more detailed exposition is given in [Mor, Sect. 6,7]. We begin by recalling the following definition from commutative algebra (see also [Mor, Def. 7.1]).

Definition 4.15. *An ideal $I \subset R$ of a Noetherian ring R is called* equiheighted *if all minimal prime ideals over I have the same height in R. We define the localization of an R-module M at I, to be*

$$M_I = S^{-1}M,\ \textit{where } S = \{s \in R \mid s \textit{ is a non-zero-divisor in } R/I\}.$$

Geometrically, an equiheighted ideal defines a closed subspace of $\operatorname{Spec} R$, with all irreducible components having the same codimension in $\operatorname{Spec} R$. Although not completely obvious, the two operations introduced below preserve chains of equiheighted ideals [Mor, Lemma 7.3].

Definition 4.16. *Let R be a Noetherian ring of Krull dimension n. For a chain of equiheighted ideals $\xi = (I_k \subset \cdots \subset I_0)$, with $\operatorname{ht} I_i = n - i$, we define the completion operation*

$$\mathsf{C}(R, \xi) = (\widehat{R}_{I_0}, \xi).$$

We denote by

$$\mathsf{L}(R, \xi) = (R_{I_1}, \xi'),$$

the localization operation, where ξ' is the restriction to R_{I_1} of the shifted chain of ideals given by

$$I'_i = I_{i+1}.$$

Example 4.17. *If R is a Noetherian domain of Krull dimension 1, then for every prime ideal $\mathfrak{p}$, we can consider the chain $\xi = (0 \subset \mathfrak{p})$. In this case, we have $(\mathsf{L} \circ \mathsf{C})(R, \xi) = \operatorname{Frac} \widehat{R}_{\mathfrak{p}}$.*

Lemma 4.18. *Let R be an excellent reduced ring of Krull dimension n. For a chain of radical equiheighted ideas $(0 = I_n \subset I_{n-1} \subset \cdots \subset I_0)$, with $\mathrm{ht} I_i = n - i$, we have*

$$F_{\operatorname{Spec} R, \xi} \cong (\mathsf{L} \circ \mathsf{C})^n (R, \xi).$$

Following Beilinson, we will also use the notation $\mathbb{A}_X(\{\xi\}, \mathcal{O}_X)$.

4.2.3. *Symbols at Flags of Subschemes.* Let X be an excellent scheme of dimension n, and

$$\xi = (X \supset Z_n \supset \cdots \supset Z_0)$$

a flag of irreducible closed subschemes. We introduce the following shorthand notation for the ring of adèles

$$F_{X,\xi} = \mathbb{A}_X(\{\xi\}, \mathcal{O}_X) \tag{19}$$

in the sense of Beilinson, as discussed in Paragraph 4.2.2. We recall Theorem 7.11 of [BGW14b], in which the authors endowed the objects $\mathbb{A}_{X,\xi}(\mathcal{F})$ inductively with the structure of higher Tate objects. In particular, we see that $F_{X,\xi}$ gives rise to an n-Tate object $\underline{F}_{X,\xi}$ in the abelian category $\mathrm{Coh}_{Z_0}(X)$ (coherent sheaves on X, set-theoretically supported at Z_0). If X is defined over a field k, then, because Z_0 is 0-dimensional, global sections give rise to an exact functor

$$\Gamma(X, -)\colon \text{ n-}\mathsf{Tate}(\mathrm{Coh}_{Z_0} X) \longrightarrow \text{n-}\mathsf{Tate}(k).$$

Thus, $F_{X,\xi}$ gives rise to an n-Tate object in the category of finite-dimensional vector spaces over k. If A is an arbitrary k-algebra, the tensor product

$$- \otimes_k A\colon \mathsf{Vect}_f(k) \longrightarrow P_f(A)$$

determines an exact functor

$$-\widehat{\otimes}_k A\colon \text{ n-}\mathsf{Tate}(k) \longrightarrow \text{n-}\mathsf{Tate}(A).$$

Definition 4.19. *Let X, ξ, k, and A be as described earlier. We define*

$$\underline{A}_{X,\xi} = F_{X,\xi} \widehat{\otimes}_k A.$$

The A-module underlying $\underline{A}_{X,\xi}$ (via the forgetful functor $\text{n-}\mathsf{Tate}(A) \longrightarrow \mathsf{Mod}(A)$) inherits a k-algebra structure from $F_{X,\xi}$; we denote this k-algebra by $A_{X,\xi}$.

For a group scheme G over k, we define the iterated loop group at (X, ξ) to be the group-valued sheaf given by

$$L^n_{X,\xi} G(A) = G(A_{X,\xi}).$$

By definition, we have $L^n_{X,\xi}\,\mathbb{G}_m = F_{X,\xi}$. However, equation (19) is also meaningful for a scheme X of mixed characteristic, where the rings $A_{X,\xi}$ cannot be defined.

Example 4.20. *If $X = \mathbb{A}^n_k = \operatorname{Spec} k[t_1, \ldots, t_n]$, and $Z_k = \operatorname{Spec} k[t_1, \ldots, t_k]$, then we have $A_{X,\xi} = A((t_1)) \ldots ((t_n))$, and $L^n_{X,\xi}\,\mathbb{G}_m \cong L^n\,\mathbb{G}_m$.*

As we have seen in Remark 4.3, the ring $A_{X,\xi}$ is non-canonically isomorphic to a product of rings $A((t_1))\dots((t_n))$, possibly tensored by a finite field extension of k. Hence the above example provides an adequate picture of the general behaviour.

Note that for any ring R, the exact category of finitely-generated projective modules $P_f(R)$ is the idempotent completion of the exact category of finitely-generated free R-modules. Therefore, any exact functor $\phi\colon P_f(R)\longrightarrow \mathsf{C}$, into any idempotent complete exact category C, is determined by $\phi(R)$ up to equivalence.

Definition 4.21. *Let $T\colon P_f(A_{X,\xi})\longrightarrow \text{n-}\mathsf{Tate}(A)$ be the unique functor sending $A_{X,\xi}$ to $\underline{A}_{X,\xi}$. The composition*

$$\sigma^A_{X,\xi}\colon\ \mathbb{K}_{A_{X,\xi}} \xrightarrow{T} \mathbb{K}_{\text{n-}\mathsf{Tate}(A)} \xrightarrow{(-1)^n\mathsf{Index}^n} \Sigma^n\,\mathbb{K}_A$$

will be referred to as the spectral Contou-Carrère symbol.

The notation T for the functor above is due to the fact that we are dealing with an algebraic analogue of the *Toeplitz construction*.

Replacing K-theory by G-theory (i.e. working with all coherent sheaves instead of only locally free ones), we obtain an analogous *spectrification* of the tame symbol. We leave the details to the reader.

Definition 4.22. *Let $T\colon P_f(F_{X,\xi})\longrightarrow \text{n-}\mathsf{Tate}(\mathrm{Coh}_{Z_0}(X))$ be the unique functor sending $F_{X,\xi}$ to $\underline{F}_{X,\xi}$. The composition*

$$\sigma_{X,\xi}\colon\ \mathbb{K}_{F_{X,\xi}} \xrightarrow{T} \mathbb{K}_{\text{n-}\mathsf{Tate}(\mathrm{Coh}_{Z_0}(X))} \xrightarrow{(-1)^n\mathsf{Index}^n} \Sigma^n\mathbb{G}_{X,Z_0} \xrightarrow{\cong} \Sigma^n\,\mathbb{K}_{Z_0} \longrightarrow \Sigma^n\,\mathbb{K}_k$$

will be referred to as the spectral tame symbol.

Switching back to the Contou-Carrère setup, we can use higher commutators to extract Contou-Carrère symbols from the morphism of spectra in Definition 4.21.

Definition 4.23. *Denote by* $\det\colon K_1(A)\longrightarrow A^\times$ *the map induced by the determinant of matrices. For an $n+1$-tuple $f_0,\dots,f_n\in A^\times_{X,\xi}$ we define*

$$(f_0,\dots,f_n)=\det((f_0\star\cdots\star f_n)_{\sigma^A_{X,\xi}}),$$

and refer to this expression as the Contou-Carrère symbol of X at ξ.

4.3. Symbols as Boundary Maps.

4.3.1. *The Calculus of Zigzags.* In this subsection we introduce zigzags, which are a mnemonic device to study composition of boundary maps in algebraic K-theory. We begin with the definition of a *basic zigzag*.

Definition 4.24. *A* basic zigzag $\mathcal{Z}$ *is a diagram in the category of pairs of schemes*

$$\begin{tikzcd} X, Z\cap W \arrow[dr, dash] & & U, W\cap U \arrow[dl, dash] \\ & X, Z & \end{tikzcd}$$

where X is a scheme, W and Z are closed subschemes, and $U = X\setminus Z$.

The concatenation of these building blocks yields a class of diagrams called *zigzags*.

Definition 4.25. *A chain of n basic zigzags $\mathcal{Z}_1,\dots,\mathcal{Z}_n$*

$$\begin{tikzcd} X_0, Z_0\cap W_0 \arrow[dr, dash] & & U_0, W_0\cap U_0 \arrow[dl, dash] \arrow[dr, dash] & \cdots & U_{n-1}, W_{n-1}\cap U_{n-1} \arrow[dl, dash] \\ & X_0, Z_0 & & X_1, Z_1 \quad \cdots & \end{tikzcd}$$

is called a zigzag $\mathcal{Z}$ of length n. We will say that $\mathcal{Z}$ is a concatenation of the basic zigzags $\mathcal{Z}_1,\dots,\mathcal{Z}_n$, and write $\mathcal{Z}=\mathcal{Z}_1\star\cdots\star\mathcal{Z}_n$.

Applying the functor $\mathbb{K}_{-}$ to a basic zigzag as in Definition 4.24, we obtain an instance of Thomason–Trobaugh's distinguished triangle ([TT90, Thm. 7.4])

$$\begin{tikzcd}
\mathbb{K}_{X,Z\cap W} \arrow[dr] & & \mathbb{K}_{U,W\cap U} \arrow[ll, dotted, "\bullet"'] \\
& \mathbb{K}_{X,Z}\,. \arrow[ur] &
\end{tikzcd}$$

The dotted arrow is the boundary map, the decoration $\bullet$ indicates that this is really a morphism of degree 1

$$\Omega\,\mathbb{K}_{U,W\cap U} \longrightarrow \mathbb{K}_{X,Z\cap W}\,.$$

Given a zigzag as in Definition 4.25, we can compose the induced boundary maps

$$\begin{tikzcd}
\mathbb{K}_{X_0,Z_0\cap W_0} \arrow[dr] & & \mathbb{K}_{U_0,W_0} \cap \mathbb{K}_{U_0} \arrow[ll, dotted, "\bullet"'] \arrow[dr] & \cdots \arrow[l, dotted, "\bullet"'] & & \mathbb{K}_{U_{n-1},W_{n-1}\cap U_{n-1}} \arrow[ll, dotted, "\bullet"'] \\
& \mathbb{K}_{X_0,Z_0} \arrow[ur] & & \mathbb{K}_{X_1,Z_1} & \cdots \arrow[ur] &
\end{tikzcd}$$

and obtain the morphism of spectra defined below.

Definition 4.26. *The composition of boundary maps associated to a zigzag $\mathcal{Z}$ will be denoted by*

$$\partial_{\mathcal{Z}}\colon \Omega^n\,\mathbb{K}_{U_{n-1},W_{n-1}\cap U_{n-1}} \longrightarrow \mathbb{K}_{X_0,Z_0\cap W_0}$$

and referred to as the iterated boundary map *associated to $\mathcal{Z}$.*

Zigzags give rise to a category in a natural way. A morphism of zigzags $f\colon \mathcal{Z} \longrightarrow \mathcal{Z}'$ exists only if $\mathcal{Z}$ and $\mathcal{Z}'$ are of the same length, in this case it is given by a compatible system of morphisms of schemes.

As an immediate consequence of the naturality of boundary maps, we have the following.

Lemma 4.27. *Let $f\colon \mathcal{Z} \longrightarrow \mathcal{Z}'$ be a morphism of zigzags. There is a commutative diagram of spectra*

$$\begin{tikzcd}
\Omega^n\,\mathbb{K}_{U'_{n-1},W'_{n-1}\cap U'_{n-1}} \arrow[r, "\partial_{\mathcal{Z}'}"] \arrow[d, "f^*"] & \mathbb{K}_{X'_0,Z'_0\cap W'_0} \arrow[d, "f^*"] \\
\Omega^n\,\mathbb{K}_{U_{n-1},W_{n-1}\cap U_{n-1}} \arrow[r, "\partial_{\mathcal{Z}}"] & \mathbb{K}_{X_0,Z_0\cap W_0}
\end{tikzcd}$$

relating iterated boundary maps with the pullback functor.

4.3.2. *The Local Zigzag Construction.* In order to define the *local zigzag* we need to introduce the following operators corresponding to completion at and removal of closed subschemes. We begin by giving a precise definition of the completion of a scheme at a closed subscheme. Although this seems fairly straight-forward in the affine case, it is necessary to be finical in general.

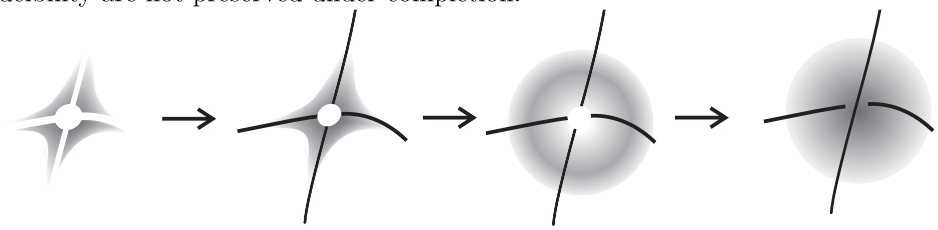
Definition 4.28. *Let X be a scheme and $\mathcal{I} \subset \mathcal{O}_X$ a sheaf of ideals for which the corresponding closed subscheme $Y \subset X$ is affine. We define the* completion of X at Y *to be the spectrum*

$$\mathbf{C}_Y\,X = \operatorname{Spec} \varprojlim_{n\in\mathbb{N}} \Gamma(X, \mathcal{O}_X/\mathcal{I}^n).$$

A related construction is the formal neighbourhood $\widehat{X}_Y$. It is defined to be the direct limit in the sense of formal schemes, of the family of schemes $Y_n = \operatorname{Spec}_X \mathcal{O}_X/\mathcal{I}^n$. The completion of Definition 4.28 on the other hand is equivalent to the direct limit of the X-schemes Y_n in the category of affine schemes.

Warning 4.29. *The above definition makes sense without assuming that Y is affine, but it may be pathological. For example, if $Y \subset \mathbb{P}^n$ is an embedded projective curve, the inverse limit $\varprojlim_{n\in\mathbb{N}} \mathcal{O}_X/\mathcal{I}^n$ in the category of $\mathcal{O}_X$-modules is not necessarily quasi-coherent.*

By virtue of Chevalley's theorem (cf. Conrad [Con07]), affineness of Y only depends on the underlying subspace $|Y| \subset |X|$ (i.e. Chevalley's theorem implies that a scheme is affine if and only if the underlying reduced scheme is affine). Subsequently we introduce notation facilitating completion and removal along a flag of closed subschemes. The picture below serves as a reminder that notions like irreducibility are not preserved under completion.



Definition 4.30. *Let (X, ξ) be a pair consisting of a scheme and a flag of closed affine subschemes $\xi\colon |X| \supset Z_{n-1} \supset \cdots \supset Z_0$, where n is an* arbitrary *integer. We define $\mathbf{C}_j(X, \xi)$ to be the pair $(\mathbf{C}_{Z_j} X, \mathbf{C}_j\,\xi)$, with $(\mathbf{C}_j\,\xi)_i = \mathbf{C}_{Z_j} Z_i$. We abbreviate $\mathbf{C}_{j+k-1}\cdots\mathbf{C}_j(X,\xi)$ by $\mathbf{C}_j^k(X,\xi)$.*

Similarly, the removal operator $\mathbf{R}$ will be applied iteratively. The following definition merely serves the purpose of fixing notation.

Definition 4.31. *The pair $\mathbf{R}_j(X,\xi)$ is defined to be $(X\backslash Z_j, \mathbf{R}\,\xi)$, with $(\mathbf{R}\,\xi)_i = Z_i\backslash Z_j$. We abbreviate $\mathbf{R}_{j+k-1}\cdots\mathbf{R}_j(X,\xi)$ by $\mathbf{R}_j^k(X,\xi)$.*

Definition 4.32. *We use the notation $(\mathbf{R}\circ\mathbf{C})_j^k(X,\xi)$, to denote*

$$\mathbf{R}_{j+k-1}\,\mathbf{C}_{j+k-1}\cdots\mathbf{R}_j\,\mathbf{C}_j(X,\xi).$$

Definition 4.33. *Let $\mathbf{Op}$ be the composition of an arbitrary sequence of the operators $\mathbf{R}$ and $\mathbf{C}$, and $A \subset |X|$ a closed subscheme. If well-defined (i.e. if the affineness condition of Definition 4.28 holds when it should), we denote by $\mathbf{Op}\,(X, A)$ the pair consisting of $\mathbf{Op}\,X$ and the closed subscheme given by $\mathbf{Op}\,A$.*

We can now introduce the local zigzag construction.

Definition 4.34. *Let X be a scheme, and $\xi = (X \supset Z_n \supset \cdots \supset Z_0)$ a flag of closed subschemes of X. We say that ξ is* admissible *if, for each j, the scheme $\mathbf{C}_j(\mathbf{R}\circ\mathbf{C})_0^j X$ is well-defined (i.e. $(\mathbf{R}\circ\mathbf{C})_0^j Z_j$ is affine for each j), and the inclusion*

$$(\mathbf{R}\circ\mathbf{C})_0^j Z_j \hookrightarrow \mathbf{C}_{j-1}(\mathbf{R}\circ\mathbf{C})_0^{j-1} Z_j$$

is affine. The local zigzag *$\mathcal{Z}_{X,\xi}^{loc}$ is the concatenation $\mathcal{Z}_1^{loc}\star\cdots\star\mathcal{Z}_n^{loc}$, where $\mathcal{Z}_j^{loc}$ is the basic zigzag*

$$\begin{tikzcd}[column sep=tiny] (\mathbf{R}\circ\mathbf{C})_0^{j-1}\,\mathbf{C}_0(X, Z_{j-1}) \arrow[dr, dash] & & (\mathbf{R}\circ\mathbf{C})_0^{j}\,\mathbf{C}_0(X, Z_j) \arrow[dl, dash] \\ & \mathbf{C}_{j-1}(\mathbf{R}\circ\mathbf{C})_0^{j-1}(X, Z_j) & \end{tikzcd} \tag{20}$$

Below we give an example of a local zigzag for $X = \mathbb{A}_R^2$, i.e. affine space over a ring R. We use the flag ξ given by the origin and the embedding of $\mathbb{A}_R^1$ as one of the coordinate axes.

Example 4.35. *Let X be affine 2-space $\mathbb{A}^2_R = \operatorname{Spec} R[s,t]$ relative to the base ring R. We consider the flag of closed subschemes given by $\mathbb{A}^2_R \supset \mathbb{A}^1_R = \operatorname{Spec} R[s] \supset \{0\}$. One obtains the following zigzag:*

$$\begin{tikzcd}[column sep=tiny]
R[[s,t]],(s,t) \arrow[dr, dash] & & R[[s,t]] \setminus (s,t), (t) \setminus (s,t) \arrow[dl, dash] \arrow[dr, dash] & & R((s))((t)). \arrow[dl, dash] \\
& R[[s,t]],(t) & & R((s))[[t]] &
\end{tikzcd}$$

For the sake of readability we have omitted the functor Spec*, and replaced closed subsets by the corresponding ideals.*

We conclude this section with an observation, which takes care of the admissibility criterion for all flags ξ of interest to us.

Lemma 4.36. (a) *Let X be an n-dimensional Noetherian scheme, and $\xi = (X \supset Z_{n-1} \supset \cdots \supset Z_0)$ a* saturated *flag, i.e. each Z_i is of pure dimension i. Then the flag ξ is admissible in the sense of Definition 4.34.*

(b) *Consider a k-algebra A, where k is a field. For a Noetherian k-scheme X of dimension n, and a saturated flag ξ, we denote by $X_A = X \times_{\operatorname{Spec} k} \operatorname{Spec} A$ and ξ_A the respective base changes along $\operatorname{Spec} A \longrightarrow \operatorname{Spec} k$. The flag ξ_A is admissible in the sense of Definition 4.34.*

Proof. We observe that $\dim Z_0 = 0$, i.e. Z_0 is a disjoint union of finitely many closed points and therefore affine. Similarly, each $(\mathbf{R} \circ \mathbf{C})_0^j Z_j$ is of dimension 0. This concludes the proof of statement (a). Assertion (b) is proven by similar means. The base change $(Z_0)_A = Z_0 \times_{\operatorname{Spec} k} \operatorname{Spec} A$ of the affine scheme along the affine morphism $\operatorname{Spec} A \longrightarrow \operatorname{Spec} k$ is again affine. The general case follows by induction, if $(\mathbf{R} \circ \mathbf{C})_0^j (Z_j)_A$ is known to be affine, then so is the completion $\mathbf{C}_j (\mathbf{R} \circ \mathbf{C})_0^j (Z_{j+1})_A$. Removing $(\mathbf{R} \circ \mathbf{C})_0^j (Z_j)_A$, i.e. considering $(\mathbf{R} \circ \mathbf{C})_0^{j+1} (Z_j)_A$ is equivalent to the base change

$$\mathbf{C}_j (\mathbf{R} \circ \mathbf{C})_0^j (Z_{j+1})_A \times_{\mathbf{C}_j (\mathbf{R} \circ \mathbf{C})_0^j Z_{j+1}} (\mathbf{R} \circ \mathbf{C})_0^{j+1} Z_{j+1}.$$

Since affineness of schemes and morphisms is preserved by base change, this concludes the argument. □

4.3.3. *Contou-Carrère Symbols and Realization Functors.* We recall the following result from Thomason–Trobaugh [TT90, Porism 2.7.1].

Lemma 4.37. *Let X be a scheme of finite type over k, with a subscheme Z finite over k (in particular $\dim Z = 0$). For every k-algebra A, we denote by*

$$\pi \colon X_A = X \times_k \operatorname{Spec} A \longrightarrow \operatorname{Spec} A$$

the canonical projection. If $\mathcal{F} \in \operatorname{Perf}_{Z_A}(X_A)$, then $\pi_* \mathcal{F}$ is a perfect complex of A-modules.

Proof. This is a special case of Porism 2.7.1 in [TT90]. Up to change of notation, the latter considers a finitely presented map $h \colon X \longrightarrow W$, a quasi-compact open subset $U \subset X$, which is the complement of a closed immersion $Z \longrightarrow X$, such that $h|_Z$ is proper, and $h|_U$ is flat. Under these assumptions it is shown that the pushforward $h_* \mathcal{F}$ of a perfect complex $\mathcal{F}$ supported on $|Z|$, is perfect.

In order to apply this result, one observes that a morphism of finite type over a field is finitely presented. Moreover, being of finite presentation, flat, or proper, is a notion invariant under base change. Since every finite morphism is in particular proper, all the conditions of the porism cited above are met. □

Using the zigzag notation introduced earlier, we are able to state the main result of this section.

Theorem 4.38. *Let X be a Noetherian k-scheme, and ξ a saturated flag of closed subschemes Z_i. For every k-algebra A, we have a projection $\pi \colon X_A \longrightarrow \operatorname{Spec} A$. The pushforward π_* sends $\operatorname{Perf}_{Z_A}(X_A)$ to $\operatorname{Perf}(A)$. Hence, we have a well-defined map*

$$\pi_* \circ \partial_{\mathcal{Z}^{loc}_{X,\xi}} \colon \Omega^n \, \mathbb{K}_{A_{X,\xi}} \longrightarrow \mathbb{K}_A \,.$$

It is equivalent to the spectral Contou-Carrère symbol

$$\pi_* \circ \partial_{\mathcal{Z}^{loc}_{X,\xi}} \simeq \sigma^A_{X,\xi}.$$

The proof of this result is based on the concept of realization functors and will be given at the end of this paragraph.

Definition 4.39 (Tate realization)**.** *Let X be a Noetherian scheme, and $j\colon U \hookrightarrow X$ an open immersion, with complement denoted by Z. Let $W \supset Z$ be a closed subscheme of X, such that the open immersion $U \cap W \hookrightarrow W$ is affine.*[4] *Then, we have exact functors* ind, pro, *and* tate, *defined as follows.*

(a) *The functor* ind: $\mathrm{Coh}_{|W|\cap U}(U) \longrightarrow \mathsf{Ind}^a(\mathrm{Coh}_{|W|}(X))$ *sends $\mathcal{F} \in \mathrm{Coh}_{|W|\cap U}(U)$ to $j_*\mathcal{F}$, viewed as an ascending union of coherent sheaves on X with set-theoretic support in $|W|$.*

(b) *We denote by $i_n\colon Z^{(n)} \longrightarrow X$ the inclusion of the n-th order infinitesimal neighbourhood of Z. We define* pro: $\mathrm{Coh}_W(X) \longrightarrow \mathrm{Coh}_{|Z|}(X)$ *to be the functor sending $\mathcal{F} \in \mathrm{Coh}_{|W|}(X)$ to the Pro-system $((i_n)_* i_n^* \mathcal{F})_{n\in\mathbb{N}}$.*

(c) *Combining (a) and (b) we obtain a functor*

$$\mathrm{tate}\colon\ \mathrm{Coh}_{|W|}(U) \longrightarrow \mathsf{Ind}^a\mathsf{Pro}^a(\mathrm{Coh}_{|Z|}(X)).$$

Remark 4.40. *One can check that the functor* tate *of Definition 4.39(c) factors through $\mathsf{Tate}^{el}(\mathrm{Coh}_Z(X)) \subset \mathsf{Ind}^a\mathsf{Pro}^a(\mathrm{Coh}_Z(X))$. Indeed, for every $\mathcal{F} \in \mathrm{Coh}_{|W|\cap U}(U)$ we have a 4-term exact sequence*

$$\ker(\mathrm{pro}(\mathcal{F}) \longrightarrow \mathrm{tate}(\mathcal{F})) \hookrightarrow \mathrm{pro}(\mathcal{F}) \longrightarrow \mathrm{tate}(\mathcal{F}) \twoheadrightarrow j_*j^*\mathcal{F}\,/\,\mathcal{F}\,.$$

The kernel on the left hand side is equivalent to $\ker(\mathcal{F} \longrightarrow j_*j^*\mathcal{F}) \in \mathrm{Coh}_Z(X)$, hence the quotient of $\mathrm{pro}(\mathcal{F})$ by this object lies again in $\mathsf{Pro}^a(\mathrm{Coh}_Z(X))$. The object on the right hand side $j_*j^*\mathcal{F}\,/\,\mathcal{F}$ lies in $\mathsf{Ind}^a(\mathrm{Coh}_Z(X))$. This allows us to represent $\mathrm{tate}(\mathcal{F})$ as an extension of an admissible Ind-object by an admissible Pro object. Hence, $\mathrm{tate}(\mathcal{F})$ has a lattice, i.e. is an elementary Tate object.

Definition 4.39 contains the condition that the inclusion $U \cap W \hookrightarrow W$ is affine. It is important to note the following two observations.

Lemma 4.41. (a) *Let $W, W' \hookrightarrow X$ be closed subschemes of a separated Noetherian scheme X, satisfying $|W| = |W'|$. For an open subscheme $U \subset X$ we have that $W \cap U \hookrightarrow W$ is affine if and only if $W' \cap U \hookrightarrow W'$ is affine.*

(b) *Let $W \hookrightarrow X$ be a closed immersion into a separated Noetherian scheme X, and $U \subset X$ an open subscheme with closed complement denoted by $|Z|$. If $|Z| \subset |W|$, and $\dim Z = \dim W - 1 = 0$, then the inclusion $W \cap U \hookrightarrow W$ is* affine.

Proof. Assertion (a) follows from the fact that a scheme is affine if and only if the underlying reduced scheme is affine (which is a consequence of Chevalley's theorem, see [Con07]). To verify (b), we observe that $Z = \{z_1, \ldots, z_k\}$ is a discrete subset consisting of closed points (since it is of dimension 0), and we may therefore replace W without loss of generality by $\mathrm{Spec}(\mathcal{O}_{W,z_1} \times \cdots \times \mathcal{O}_{W,z_k})$. Then, the complement $W \cap U = W \setminus Z$ agrees with the discrete subset $\{\eta_1, \ldots, \eta_m\}$, where each η_i is a generic point of the one-dimensional scheme W. Since each of the inclusions $\{\eta_i\} \hookrightarrow X$ is affine, the same is true for

$$W \cap U = \coprod_{i=1}^m \{\eta_i\} \hookrightarrow X,$$

since a finite coproduct of affine schemes is affine. □

Assumption 4.42. *We assume that our schemes satisfy the assumption of Thomason–Trobaugh's localization theorem, i.e. are quasi-compact and quasi-separated.*

We call these functors *realization functors*, since they associate to a coherent sheaf on $U = X \setminus Z$ a Tate object in $\mathrm{Coh}_Z(X)$. For our purposes it will be necessary to have similar functors for *perfect complexes* on U at our disposal. This is achieved by the following definition.

[4] In Lemma 4.41(a) we show that this is condition only depends on the underlying closed subspace $|W|$.

Definition 4.43 (Calkin realization)**.** *Let X, U, Z, and W be quasi-compact and quasi-separated schemes, with $Z \subset W$, and $U = X \setminus Z$.*

(a) *We denote by*

$$\mathrm{ind}\colon \mathrm{Perf}(X) \longrightarrow \mathsf{Ind}(\mathrm{Perf}_Z(X)) \cong \mathrm{QCoh^{der}}_Z(X)$$

the functor given by $\mathcal{F} \mapsto \mathrm{fib}(\mathcal{F} \longrightarrow j_*j^*\mathcal{F})$.

(b) *Let* $\mathrm{calk}\colon \mathrm{Perf}(U) \longrightarrow \mathsf{Calk}(\mathrm{Perf}_Z(X))$ *be the functor induced by* ind*:*

$$(\mathrm{Perf}(X)/\mathrm{Perf}_Z(X))^{\mathrm{ic}} \longrightarrow (\mathsf{Ind}(\mathrm{Perf}_Z(X))/\mathrm{Perf}_Z(X))^{\mathrm{ic}}.$$

(c) *The functors* ind *and* calk *have a version for the stable ∞-categories of pseudo-coherent complexes of sheaves:*

$$\mathrm{ind}\colon \mathrm{Coh}^{der}(X) \longrightarrow \mathsf{Ind}(\mathrm{Coh}^{der}_Z(X)),$$

and

$$\mathrm{calk}\colon \mathrm{Coh}^{der}(U) \longrightarrow \mathsf{Calk}(\mathrm{Coh}^{der}_Z(X)).$$

The Calkin realization is evidently compatible with base change along *flat maps*, which preserve Assumption 4.42 (e.g., affine flat morphisms).

Lemma 4.44. *Let X, U, and Z be as in Definition 4.43. Let $X \longrightarrow W$ be a morphism of schemes, with W satisfying Assumption 4.42. For an affine flat morphism $V \longrightarrow W$ we denote the base changes $X \times_W V$, $U \times_W V$, and $Z \times_W V$ by X_V, U_V, and Z_V. We then have a commutative diagram*

$$\begin{tikzcd}
\mathrm{Coh}^{der}(U) \arrow[r] \arrow[d] & \mathsf{Calk}(\mathrm{Coh}^{der}_Z(X)) \arrow[d] \\
\mathrm{Coh}^{der}(U_V) \arrow[r] & \mathsf{Calk}(\mathrm{Coh}^{der}_{Z_V}(X_V)).
\end{tikzcd}$$

The Tate and Calkin realization for coherent sheaves (respectively pseudo-coherent complexes) are related by the natural exact functor

$$q\colon \mathsf{Tate}(\mathrm{Coh}_Z(X)) \longrightarrow \mathsf{Calk}(\mathrm{Coh}_Z(X)) \longrightarrow \mathsf{Calk}(\mathrm{Coh}^{der}_Z(X)).$$

Lemma 4.45. *Let X be Noetherian, and Z, U, and W satisfy the conditions of Definition 4.39. We have a commuting square*

$$\begin{tikzcd}
\mathrm{Coh}_{|W|\cap U}(U) \arrow[r] \arrow[d] & \mathsf{Tate}(\mathrm{Coh}_{|Z|}(X)) \arrow[d, "{q[-1]}"] \\
\mathrm{Coh}^{der}_{|W|\cap U}(U) \arrow[r] & \mathsf{Calk}(\mathrm{Coh}^{der}_{|Z|}(X)).
\end{tikzcd}$$

Proof. According to Definition 4.43 we have that, for every pseudo-coherent complex $\mathcal{F}$ on X with set-theoretic support in $|W|$,

$$\mathrm{calk}(j^*\mathcal{F}) = \mathrm{fib}(\mathcal{F} \longrightarrow j_*j^*\mathcal{F}).$$

Since j is flat and affine by assumption, we have that for $\mathcal{F} \in \mathrm{Coh}(X)$ the expression $j_*j^*\mathcal{F}$ has vanishing higher cohomology groups. In particular, $\mathrm{calk}(j^*\mathcal{F})$ can be represented by the admissible Ind-object $j_*j^*\mathcal{F}/\mathcal{F}[-1]$. By Remark 4.40, this admissible Ind-object represents the Calkin object corresponding to the Tate object $\mathrm{tate}(\mathcal{F})$. The general case, i.e. of a coherent sheaf on U which does not extend to X, follows by passing to idempotent completions.

The discussion above gives rise to the top square in the commutative diagram below:

$$\begin{tikzcd}
\mathrm{Coh}_{|W|\cap U}(U) \arrow[r] \arrow[d] & \mathsf{Tate}(\mathrm{Coh}_{|Z|}(X)) \arrow[d] \\
\mathrm{Coh}_{|W|\cap U}(U) \arrow[r, "\phi"] \arrow[d] & \mathsf{Calk}(\mathrm{Coh}_{|Z|}(X)) \arrow[d] \\
\mathrm{Coh}^{der}_{|W|\cap U}(U) \arrow[r, "\mathrm{calk}[1]"] & \mathsf{Calk}(\mathrm{Coh}^{der}_{|Z|}(X)),
\end{tikzcd}$$

where ϕ is the functor obtained by sending $\mathcal{F} \in \mathrm{Coh}^{der}_{|W|\cap U}(U)$ to $j_*\mathcal{F}/\widetilde{F} \in \mathsf{Calk}(\mathrm{Coh}^{der}_{|Z|}(X))$, where $\widetilde{\mathcal{F}}$ is a pseudo-coherent subsheaf of $j_*\mathcal{F}$, such that $j^*\widetilde{F} = \mathcal{F}$. The outer square yields the required commutative diagram. □

If X is a Noetherian scheme, and ξ a saturated flag of closed subschemes, then for each basic constituent (20) of the local zigzag of (X, ξ) we will see that we have a Tate realization functor (recall the convention from Definition 4.33)

$$\mathrm{tate}\colon\ \mathrm{Coh}((\mathbf{R}\circ\mathbf{C})^j_0(X, Z_j)) \longrightarrow \mathrm{Coh}((\mathbf{R}\circ\mathbf{C})^{j-1}_0(X, Z_{j-1})).$$

Lemma 4.41(b) implies that the crucial affineness condition of Definition 4.39 is satisfied for dimension reasons. Composition of these exact functors yields a well-defined exact functor

$$\mathrm{tate}^n\colon\ \mathrm{Coh}((\mathbf{R}\circ\mathbf{C})^n X) \longrightarrow \text{n-}\mathsf{Tate}(\mathrm{Coh}_{Z_0}(X)).$$

Proposition 4.46. *The functor $\pi_*\,\mathrm{tate}^n$ agrees with the n-Tate object valued Beilinson-Parshin adèles $\mathbb{A}_X(\{\xi\}, -)$ of* [BGW14b].

Proof. The functors ind and pro of Definition 4.39 mirror localization and completion with respect to the scheme X. In particular, we see for $\mathcal{F} \in \mathrm{Coh}(W)$ that $\pi_*\,\mathrm{tate}^n(\mathcal{F}) = \mathbb{A}_X(\{\xi\}, \mathcal{F})$. □

Composing n-Tate with pushforward $\pi\colon\ \mathrm{Coh}_{Z_0}(X) \longrightarrow \mathrm{Coh}(\operatorname{Spec} k) = \mathsf{Vect}_f(k)$, we obtain an exact functor

$$\mathrm{Coh}((\mathbf{R}\circ\mathbf{C})^n X) \longrightarrow \mathsf{Tate}(k).$$

Definition 4.47. *Let X be a Noetherian k-scheme, and ξ a saturated flag of closed subschemes. For every k-algebra A we denote by $\mathrm{Coh}^\flat((\mathbf{R}\circ\mathbf{C})^n X_A)$ the full subcategory of $\mathrm{Coh}((\mathbf{R}\circ\mathbf{C})^n X_A)$, consisting of coherent sheaves which are pulled back from $(\mathbf{R}\circ\mathbf{C})^n X$. Denoting by*

$$-\widehat{\otimes}_k A\colon\ \text{n-}\mathsf{Tate}(k) \longrightarrow \text{n-}\mathsf{Tate}(A)$$

the exact functor induced by $-\otimes_k A\colon \mathsf{Vect}_f(k) \longrightarrow P_f(A)$, we have a unique A-linear functor

$$(\pi_*\circ\mathrm{tate}^n)_A\colon\ \mathrm{Coh}^\flat((\mathbf{R}\circ\mathbf{C})^n X_A) \longrightarrow \mathit{Mod}(A),$$

such that the diagram

$$\begin{tikzcd}
\mathrm{Coh}((\mathbf{R}\circ\mathbf{C})^n X) \arrow[r] \arrow[d] & \mathsf{Vect}(k) \arrow[d] \\
\mathrm{Coh}^\flat((\mathbf{R}\circ\mathbf{C})^n X_A) \arrow[r] & \mathit{Mod}(A)
\end{tikzcd}$$

commutes.

Proposition 4.48. *We denote by $\mathrm{VB}_f(W)$ the exact category of* free vector bundles *on a scheme W. Let X be a finite type, separated k-scheme of dimension n, and let ξ be a saturated flag of closed*

subschemes. For every k-algebra A we have a commutative square

$$\begin{tikzcd} \mathrm{VB}_f((\mathbf{R}\circ\mathbf{C})^n X_A) \arrow[r, "(\pi_* \operatorname{tate}^n)_A"] \arrow[d] & \text{n-}\mathsf{Tate}(A) \arrow[d, "{q[-n]}"] \\ \mathrm{Perf}((\mathbf{R}\circ\mathbf{C})^n X_A) \arrow[r, "\pi_* \operatorname{calk}^n"] & \mathsf{Calk}^n(\mathrm{Perf}(A)). \end{tikzcd}$$

Proof. For $A = k$ this follows from applying the comparison of Lemma 4.45 iteratively. The general case follows from the base change invariance of the Calkin realization (Lemma 4.44), and Definition 4.47 of the functor $(\pi_* \circ \operatorname{tate}^n)_A$ by base change. □

We are now ready to prove that the spectral Contou-Carrère symbol $\sigma^A_{X,\xi}$ can be represented as the composition $\pi_* \circ \partial_{\mathcal{Z}^{\mathrm{loc}}_{X_A,\xi_A}}$.

Proof of Theorem 4.38. Proposition 4.48 established a compatibility between the Tate and Calkin realization:

$$\pi_* \operatorname{calk}^n \simeq q(\pi_* \operatorname{tate}^n)_A[-n].$$

Applying the non-connective algebraic K-theory functor $\mathbb{K}_-$ to this equivalence, we obtain to equivalent maps

$$\mathbb{K}_{\pi_* \operatorname{calk}^n} \simeq \mathbb{K}_{q(\pi_* \operatorname{tate}^n)_A[-n]} \colon \mathbb{K}_{(\mathbf{R}\circ\mathbf{C})^n X_A} \longrightarrow \mathbb{K}_{\mathsf{Calk}^n(\mathrm{Perf}(A))} . \tag{21}$$

Here, we make use of the fact that non-connective algebraic K-theory is cofinal invariant, i.e. cannot distinguish between an exact category and its idempotent completion. In particular,

$$\mathbb{K}_{\mathrm{VB}_f((\mathbf{R}\circ\mathbf{C})^n X_A)} \cong \mathbb{K}_{\mathrm{VB}((\mathbf{R}\circ\mathbf{C})^n X_A)} \cong \mathbb{K}_{(\mathbf{R}\circ\mathbf{C})^n X_A} .$$

We have commutative squares

$$\begin{tikzcd} \mathbb{K}_{\mathsf{Calk}^n(P_f(A))} \arrow[r, "\partial^n"] \arrow[d, "\cong"'] & \Sigma^n \mathbb{K}_{P_f(A)} \arrow[d, "\cong"] \\ \mathbb{K}_{\mathsf{Calk}^n(\mathrm{Perf}(A))} \arrow[r, "\partial^n"] & \Sigma^n \mathbb{K}_{\mathrm{Perf}(A)}, \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbb{K}_{\text{n-}\mathsf{Tate}(A)} \arrow[r, "\mathsf{Index}^n"] \arrow[d, "\cong"', "q"] & \Sigma^n \mathbb{K}_A \\ \mathbb{K}_{\mathsf{Calk}^n(P_f(A))} . \arrow[ur, "(-1)^n \partial^n"'] & \end{tikzcd}$$

Appending this diagram to the equivalence (21), we obtain a commutative diagram

$$\begin{tikzcd} \mathbb{K}_{(\mathbf{R}\circ\mathbf{C})^n X_A} \arrow[r, "{(\pi_* \operatorname{tate}^n)_A[-n]}"] \arrow[d, "1"'] & \Sigma^n \mathbb{K}_A \\ \mathbb{K}_{(\mathbf{R}\circ\mathbf{C})^n X_A} . \arrow[ur, "\pi_* \operatorname{calk}"'] & \end{tikzcd}$$

By Proposition 4.46, and by the definition of $(\pi_* \operatorname{tate}^n)_A$ by base change (Definition 4.47), we see that the top row agrees with the map

$$\mathbb{K}_{(\mathbf{R}\circ\mathbf{C})^n X_A} \xrightarrow{\pi_* \mathbb{A}_X(\{\xi\},-)} \mathbb{K}_{\text{n-}\mathsf{Tate}(A)} \xrightarrow{\mathsf{Index}^n} \Sigma\, \mathbb{K}_A .$$

Hence, the top row agrees with the spectral Contou-Carrère symbol, by Definition 4.21.

To conclude the proof we have to compare the bottom row with the n-fold composition of the boundary map

$$\mathbb{K}_{(\mathbf{R}\circ\mathbf{C})^n X_A} \xrightarrow{\partial_{\mathcal{Z}^{\mathrm{loc}}_{X_A,\xi_A}}} \Sigma^n \mathbb{K}_{X_A,(Z_0)_A} \xrightarrow{\pi_*} \Sigma^n \mathbb{K}_A\,.$$

Definition 4.43 implies that for every triple (X,U,Z) we have a commutative cube of stable ∞-categories

$$\begin{tikzcd}
 & 0 \arrow[rr] \arrow[dd] & & \mathrm{Perf}(U) \arrow[dd, "\mathrm{calk}"] \\
\mathrm{Perf}_Z(X) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathrm{Perf}(X) \arrow[ur] \arrow[dd] & \\
 & 0 \arrow[rr] & & \mathsf{Calk}(\mathrm{Perf}_Z(X)) \\
\mathrm{Perf}_Z(X) \arrow[ur] \arrow[rr] & & \mathsf{Ind}(\mathrm{Perf}_Z(X)). \arrow[ur] &
\end{tikzcd}$$

Since the top and bottom face are localization sequences, applying $\mathbb{K}_-$ yields a commutative cube with top and bottom face being bicartesian. In particular, we obtain a commutative triangle relating the boundary maps of the bottom and top face:

$$\begin{tikzcd}
\mathbb{K}_{\mathrm{Perf}(U)} \arrow[r, "\partial"] \arrow[d, "\mathrm{calk}"'] & \Sigma\,\mathbb{K}_{X,Z} \\
\mathbb{K}_{\mathsf{Calk}(\mathrm{Perf}_Z(X))}\,. \arrow[ur, "\partial"'] &
\end{tikzcd}$$

Applying this comparison n times, we see that $\mathbb{K}_{\pi_*\,\mathrm{calk}^n}$ is equivalent to $\pi_*\partial_{\mathcal{Z}^{\mathrm{loc}}_{X_A,\xi_A}}$. ☐

5. Derived Completion of Schemes and Categories

"You complete me." - J. Maguire

The interpretation of the tame symbol as boundary morphism in algebraic K-theory, as explained in Proposition 4.1 is ubiquitous in the classical treatments of both algebraic K-theory and the tame symbol. An analogous result for Contou-Carrère symbols has been discussed in Theorem 4.38. Before using this K-theoretic interpretation to prove reciprocity laws in Section 6, we discuss the classical constructions given by completing a ring at an ideal and Beilinson-Parshin adèles (see Paragraph 4.2.2). In order to work with an arbitrary commutative ring of coefficients A, we must study completions of non-Noetherian schemes, and, for this, we will need to treat completion together with its *derived functor*.

The study of *derived completion* goes back to work of Greenlees–May [GM92], Dwyer–Greenlees [DG02], and was embedded into the realm of derived algebraic geometry by Lurie [Lur09] and Gaitsgory–Rozenblyum [GR11]. We will mostly follow Lurie [Lur09, Ch. 4 & 5].

For every ring R, and an ideal I, we recall (see Subsection 5.1) Lurie's definition of the derived completion $\widehat{R}^{\mathrm{der}}_I$. This is a connective E_∞-ring spectrum ([Lur09, Sect. 4.2]). If R is Noetherian, the derived completion is canonically equivalent to its classical counterpart [Lur09, Prop. 4.3.6]. However, for a non-Noetherian ring R, the derived completion is genuinely different, which affects the stable ∞-category of perfect complexes.

In Subsection 5.2 we rephrase and generalize constructions of Efimov [Efi10]; we show how perfect complexes on the derived completions can be understood by an abstract construction on the level of stable ∞-categories.

We then use a calculation of Porta–Shaul–Yekutieli [PSY14] to conclude that $\widehat{R}_I^{\mathrm{der}}$ is in fact a classical ring, if I is *weakly proregular* in R (see Definition 5.3). This will allow us to remove derived rings from our work in retrospect.

5.1. **Derived Completion.** We fix a ring R and a finitely generated ideal I. We briefly review the notion of *derived complete* complexes of R-modules, as studied in [Lur09, Sect. 4.2]. A review of this material in the language of triangulated categories is given in [The, Tag 091N]. We say that a complex of R-modules is I*-complete*, if for every $x \in I$ the homotopy limit of the inverse system

$$\lim[\cdots \to M \xrightarrow{x} M \xrightarrow{x} M],$$

i.e. the fibre of

$$\prod_{n\geq 0} M \xrightarrow{x} \prod_{n\geq 1} M,$$

vanishes in the stable ∞-category $\mathsf{Mod}^{\mathrm{der}}(R)$. This is precisely the homotopical analogue of the condition that x acts *topologically nilpotently* on M, i.e.

$$\bigcap_{n\in\mathbf{N}} I^n M = 0.$$

The resulting full sub-category of I-complete objects in $\mathsf{Mod}^{\mathrm{der}}(R)$ will be denoted by

$$\mathsf{Mod}^{\mathrm{der}}(R)^{I\text{-comp}}.$$

Note that in [Lur09] this sub-category is characterized differently (cf. [Lur09, Cor. 4.2.8 & 4.2.12]). For abstract reasons, the inclusion

$$\mathsf{Mod}^{\mathrm{der}}(R)^{I\text{-comp}} \subset \mathsf{Mod}^{\mathrm{der}}(R)$$

possesses a left adjoint (see [Lur09, Lemma 4.2.2])

$$\widehat{(-)}^{\mathrm{der}}\colon \mathsf{Mod}^{\mathrm{der}}(R) \longrightarrow \mathsf{Mod}^{\mathrm{der}}(R)^{I\text{-comp}},$$

which will be referred to as *derived completion*. By Remark 4.2.6 in *loc. cit.* this is moreover a symmetric monoidal functor, hence we obtain an E_∞-ring spectrum $\widehat{R}_I^{\mathrm{der}}$; the *derived completion of R at I*.

5.2. **Modification of Stable ∞-Categories.** Let X be a scheme, Z a closed subscheme, which is defined by a locally finitely-generated sheaf of ideals. The aforementioned derived completion operation allows to define the derived formal scheme $\widehat{X}_Z^{\mathrm{der}}$ (see [Lur09, Def. 5.1.1]). If X is Noetherian, it is canonically equivalent to the formal completion $\widehat{X}_Z$. We denote by U the open complement $X \setminus Z$. Recall that $\mathrm{QCoh}^{\mathrm{der}}(X)$ denotes the stable ∞-category of complexes of quasi-coherent sheaves on X. Pullback along the open immersion $j\colon U \longrightarrow X$ induces a localization

$$j^*\colon \mathrm{QCoh}^{\mathrm{der}}(X) \longrightarrow \mathrm{QCoh}^{\mathrm{der}}(U).$$

The kernel, i.e. the full sub-category of complexes $\mathcal{F}$ satisfying $j^*\mathcal{F} \simeq 0$, will be denoted by ${\mathrm{QCoh}^{\mathrm{der}}}_Z(X)$. Since $j^*\mathcal{F} \simeq 0$ amounts to $\mathcal{F}|_U \simeq 0$, it is sensible to refer to such a complex of sheaves $\mathcal{F}$ as having *set-theoretic support contained in* $|Z|$.

The ∞-category of compact objects in ${\mathrm{QCoh}^{\mathrm{der}}}_Z(X)$ is given by $\mathrm{Perf}_Z(X)$, i.e. perfect complexes on X with set-theoretic support contained in $|Z|$. Moreover, ${\mathrm{QCoh}^{\mathrm{der}}}_Z(X)$ is *compactly generated*, amounting to the relation

$${\mathrm{QCoh}^{\mathrm{der}}}_Z(X) \cong \mathsf{Ind}\,\mathrm{Perf}_Z(X).$$

Besides passing to open subschemes (*localization* in terms of stable ∞-categories), and restricting set-theoretic support (*localizing sub-categories*), a third geometrically relevant operation is given by considering complexes of sheaves on the formal completion $\widehat{X}_Z^{\mathrm{der}}$.

Quasi-coherent sheaves on the formal completion $\widehat{X}_Z$ are closely related to the ∞-category $\mathrm{QCoh^{der}}_Z(X)$. In fact, we have an agreement of the full sub-categories of almost connective complexes ([Lur09, Thm. 5.1.9])

$$\mathrm{QCoh^{der}}(\widehat{X}_Z^{\mathrm{der}})^{\mathrm{acn}} \cong \mathrm{QCoh^{der}}_Z(X)^{\mathrm{acn}} \cong (\mathsf{Ind}\,\mathrm{Perf}_Z(X))^{\mathrm{acn}}. \tag{22}$$

Our main interest lies in the category of perfect complexes $\mathrm{Perf}(\widehat{X}_Z^{\mathrm{der}})$ on $\widehat{X}_Z^{\mathrm{der}}$. Unlike the case of a scheme, it is not sufficient to consider the full sub-category of compact objects in $\mathrm{QCoh^{der}}(\widehat{X}_Z^{\mathrm{der}})$ (denoted by upper script "c"). As we have seen earlier, $\mathrm{QCoh^{der}}(\widehat{X}_Z^{\mathrm{der}})^c \cong \mathrm{QCoh^{der}}_Z(X)^c \cong \mathrm{Perf}_Z(X)$ only yields perfect complexes with set-theoretic support contained in $|Z|$. In fact it is not very difficult to verify in the case of the structure sheaf $\mathcal{O}$ on the formal scheme $\mathrm{Spf}\ k[[t]]$ that it is not compact.

In the remainder of this subsection we will use the observations described here to develop categorical analogues of the geometric operations given by the removal of closed subschemes and completion.

5.2.1. *Completion.* Let C be an idempotent complete stable ∞-category, with a full stable sub-category $\mathbf{S}$, which is idempotent complete. We refer to such an $\mathbf{S}$ simply as *localizing* sub-category of C. Inspired by (22) we make the following definition for the *completion of* C *at* $\mathbf{S}$. Proposition 5.2 below compares this definition with the derived completion of rings.

Definition 5.1. *The completion* $\mathsf{C}_{\widehat{\mathbf{S}}}$ *is defined to be the idempotent closure of the* essential image

$$\mathrm{Im}[\mathsf{C} \longrightarrow \mathsf{Ind}\mathbf{S}]$$

of the functor sending $\mathcal{G} \in \mathsf{C}$ *to the presheaf*[5] $\mathcal{F} \mapsto \mathit{Hom}(\mathcal{F}, \mathcal{G})$.

Note that, because the inclusion $\mathbf{S} \subset \mathsf{C}$ preserves finite colimits by assumption, the presheaf associated to $\mathcal{F} \in \mathsf{C}$ preserves colimits as well, and thus yields a well-defined functor $\mathsf{C} \longrightarrow \mathsf{Ind}\mathbf{S}$.

Just like in Efimov's [Efi10, p. 8], we think of $\mathsf{C}_{\widehat{\mathbf{S}}}$ as a completion on the level of *Hom-spaces*, not altering the class of objects. The result below can be also found in [Efi10, Remark 5.3] for Noetherian rings.

Proposition 5.2. *If* $\mathsf{C} \cong \mathrm{Perf}(R)$*, where* R *is a ring, and* $\mathbf{S} = \mathrm{Perf}_{V(I)}(X)$ *for some ideal* $I \subset R$*, then* $\mathrm{Perf}(R)_{\widehat{\mathbf{S}}} \cong \mathrm{Perf}(\widehat{R}_I^{der})$.

Proof. In the following we denote by $V(I) \subset \mathrm{Spec}\, R$ the closed subset corresponding to the ideal I. We begin the proof by connecting the derived formal completion $\widehat{R}_I^{\mathrm{der}}$ of Subsection 5.1 with $\mathrm{Perf}_{V(I)}(R)$. Theorem 5.1.9 and Proposition 5.1.17 in [Lur09] imply the existence of a commutative diagram

$$\begin{tikzcd}
\mathrm{Perf}(\widehat{R}_I^{\mathrm{der}}) \arrow[rr, hook] & & \mathrm{QCoh^{der}}_{V(I)}(\mathrm{Spec}\, R) \\
& \mathrm{Perf}(R) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

of ∞-categories. Using that $\mathrm{QCoh^{der}}_{V(I)}(\mathrm{Spec}\, R)$ is compactly generated by $\mathrm{Perf}_{V(I)}(R)$, and the definition of $\mathrm{Perf}(R)_{\widehat{\mathbf{S}}}$ as the idempotent completion of the essential image of the functor

$$\mathrm{Perf}(R) \longrightarrow \mathsf{Ind}\,\mathrm{Perf}_{V(I)}(R) \cong \mathrm{QCoh^{der}}_{V(I)}(\mathrm{Spec}\, R),$$

we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Perf}(\widehat{R}_I^{\mathrm{der}}) \arrow[r, hook] & \mathsf{Ind}\,\mathrm{Perf}_{V(I)}(R) \\
\mathrm{Perf}(R) \arrow[u] \arrow[r, two heads] & \mathrm{Perf}(R)_{\widehat{\mathbf{S}}}, \arrow[u, hook] \arrow[ul, hook, dashed]
\end{tikzcd}$$

[5] Recall that $\mathsf{Ind}(\mathsf{C})$ can be realized as the ∞-category of limit-preserving functors $\mathsf{C}^{\mathrm{op}} \longrightarrow \mathsf{Sp}$.

where we use the universal property of idempotent completion to produce the dashed arrow, together with the essential surjectivity of the lower horizontal functor up to idempotent completion. In order to conclude the proof, it suffices to show that we have an inclusion

$$\mathrm{Perf}(\widehat{R}_I^{\mathrm{der}}) \subset \mathrm{Perf}(R)_{\widehat{\mathbf{S}}}$$

of full sub-categories of $\mathsf{Ind}\,\mathrm{Perf}_{V(I)}(R)$. This follows from the fact that $\mathrm{Perf}(\widehat{R}_I^{\mathrm{der}})$ is compactly generated by the structure sheaf (or free module) $\mathcal{O}$, which is contained in $\mathrm{Perf}(R)_{\widehat{\mathbf{S}}}$ by the commuting diagram above. $\square$

In the result below, we use the notion of *weak proregularity*, which was introduced by Alonso–Jeremias–Lipman [ATJLL97] and Schenzel [Sch03].

Definition 5.3. *Let R be a ring, and $f \in R$ an element. We denote by $K(R, f)$ the Koszul complex $[R \xrightarrow{\cdot f} R]$, concentrated in degrees -1 and 0. For a tuple $\overline{f} = (f_0, \dots, f_n)$ we define the Koszul complex as $K(R, \overline{f}) = \bigotimes_{i=0}^n K(R, f_i)$. An ideal $I \subset R$ is said to be* weakly proregular*, if there exist generators $(f_0, \dots, f_n)$, such that for all integers k, the inverse system of cohomology groups $\left(H^k(K(R, (f_0^i, \dots, f_n^i)))\right)_i$ is* pro-zero*, i.e. equivalent to the zero object in the category of pro-abelian groups.*

Every ideal I in a Noetherian ring is weakly proregular. Moreover, the notion of weak proregularity is evidently invariant under flat base change. Hence, if R is a Noetherian k-algebra, and A is an arbitrary k-algebra, then the ideal $I_A = I \otimes_k A \subset R_A = R \otimes_k A$ is weakly proregular.

Proposition 5.4. *If I is* weakly proregular *in R (see Definition 5.3), then $\widehat{R}_I^{der} \cong \widehat{R}_I$. In particular, we see that, for a Noetherian k-algebra R, an ideal I, and an arbitrary k-algebra A, we have $\widehat{(R_A)}_{I_A}^{der} \cong \widehat{(R_A)}_{I_A}$.*

Proof. To prove this assertion we cite the main result of Porta–Shaul–Yekutieli [PSY14, Thm. 4.2]. They prove that for every perfect generator $M \in \mathrm{Perf}_{V(I)}(A)$, the so-called *double centralizer* is equivalent to the classical formal completion $\widehat{A}$. The double centralizer of M is defined as follows. First one introduces the E_1-algebra $B = \mathsf{End}_R(M)$. The E_1-algebra $\mathsf{End}_B(M)$ is by definition the double centralizer of M.

We relate $\widehat{R}_I^{\mathrm{der}}$ to the double centralizer by observing that by definition its underlying E_1-ring agrees with the endomorphism algebra of the image of R in $\mathrm{Perf}(R)_{\widehat{V(I)}}$:

$$\widehat{R}_I^{\mathrm{der}} \cong \mathsf{End}_{\mathrm{Perf}(R)_{\widehat{V(I)}}}(R).$$

The map $\mathrm{Perf}(R) \longrightarrow \mathsf{Ind}\,\mathrm{Perf}_{V(I)}(R)$ is given by sending a module N to the presheaf $\mathsf{Hom}_R(-, N)$ on $\mathrm{Perf}_{V(I)}(R)$. Since M is a generator, we have $\mathsf{Ind}\,\mathrm{Perf}_{V(I)}(R) \cong \mathsf{Mod}^{\mathrm{der}}(\mathsf{End}_R(M)^{\mathrm{op}}) \cong \mathsf{Mod}^{\mathrm{der}}(B^{\mathrm{op}})$. In particular, we see that the R-module R is sent to

$$\mathsf{Hom}_R(M, R) = M^\vee \in \mathsf{Mod}^{\mathrm{der}}(B)^{\mathrm{op}}.$$

Thus, we have

$$\mathsf{End}_{B^{\mathrm{op}}}(M^\vee) \cong \mathsf{End}_B(M).$$

The right hand side is by definition the double centralizer of M, and therefore, by *loc. cit.* agrees with the classical completion $\widehat{R}$. In particular, since this is a discrete E_1-ring, this argument specifies the E_∞-structure as well. $\square$

Since the Yoneda embedding of $\mathbf{S}$ is fully faithful, one obtains that $\mathbf{S} \hookrightarrow \mathsf{C}_{\widehat{\mathbf{S}}}$ embeds fully faithfully into the formal completion $\mathsf{C}_{\widehat{\mathbf{S}}}$.

Definition 5.5. *Let $\mathbf{S} \subset \mathbf{T} \subset \mathsf{C}$ be a chain of localizing sub-categories of C. Then, we denote by*

(a) $\mathbf{T}_{\widehat{\mathbf{S}}} \subset \mathsf{C}_{\widehat{\mathbf{S}}}$ *the localizing sub-category given by the idempotent closure of the essential image*

$$\mathrm{Im}[\mathbf{T} \longrightarrow \mathsf{C}_{\widehat{\mathbf{S}}}],$$

and by

(b) $\mathbf{T}_{(\mathbf{S})}$ *the idempotent completion of the essential image*

$$\mathrm{Im}[\mathbf{T} \longrightarrow \mathsf{C}/\mathbf{S}].$$

As dictated by geometric intuition, completion of C at $\mathbf{S}$, followed by completion at $\mathbf{T}$, yields an ∞-category equivalent to $\mathsf{C}_{\widehat{\mathbf{S}}}$. Similarly, the completion of X at Z should be canonically equivalent to the completion of U at Z, if U is any open subscheme containing Z. This is the content of the next lemma (see also [Efi10, Thm. 4.1(iii)]).

Lemma 5.6. (a) *Using the notation of Definition 5.5, the natural map*

$$\mathsf{C}_{\widehat{\mathbf{S}}} \xrightarrow{\cong} (\mathsf{C}_{\widehat{\mathbf{S}}})_{\widehat{\mathbf{T}_{\widehat{\mathbf{S}}}}}$$

is an equivalence.

(b) *Let $\mathbf{S}, \mathbf{T}$ be localizing sub-categories, such that $\mathbf{S} \longrightarrow \mathsf{C}/\mathbf{T}$ is fully faithful (i.e. $\mathbf{S} \cap \mathbf{T} = \{0\}$). We denote by D the idempotent completion of $\mathsf{C}/\mathbf{S}$. Then we have $\mathsf{D}_{\widehat{\mathbf{T}}} \cong \mathsf{C}_{\widehat{\mathbf{T}}}$.*

Proof. (a) By definition, the right hand side agrees with the the essential image (up to idempotent completion)

$$\mathrm{Im}[\mathsf{C}_{\widehat{\mathbf{S}}} \longrightarrow \mathsf{Ind}\mathbf{T}_{\widehat{\mathbf{S}}}]^{\mathrm{ic}} \cong \mathrm{Im}[\mathrm{Im}[\mathsf{C} \longrightarrow \mathsf{Ind}\mathbf{S}]^{\mathrm{ic}} \longrightarrow \mathsf{Ind}(\mathrm{Im}[\mathbf{T} \longrightarrow \mathsf{Ind}\mathbf{S}]^{\mathrm{ic}})]^{\mathrm{ic}}.$$

The latter is equivalent to the essential image (up to idempotent completion)

$$\mathrm{Im}[\mathsf{C} \longrightarrow \mathsf{Ind}\mathbf{S}]^{\mathrm{ic}},$$

which agrees with $\mathsf{C}_{\widehat{\mathbf{S}}}$ by definition. (b) follows right from the definition. □

Definition 5.7. *For a localizing sub-category $\mathbf{S} \subset \mathsf{C}$ we denote by $\mathsf{C}_{\widehat{(\mathbf{S})}}$ the idempotent completion of the localization $\mathsf{C}_{\widehat{\mathbf{S}}}/\mathbf{S}$ of $\mathsf{C}_{\widehat{\mathbf{S}}}$ at $\mathbf{S}$.*

This localization should be imagined as the ∞-category of perfect complexes on punctured formal neighbourhood. For $X = \operatorname{Spec} R$ an affine scheme, and $Z = V(I)$ a closed subset, let $\mathbf{S} = \mathrm{Perf}_Z(X) \subset \mathrm{Perf}(X)$. We have $\mathrm{Perf}(X)_{\widehat{(\mathbf{S})}} \cong \mathrm{Perf}(\operatorname{Spec} \widehat{R}_I^{\mathrm{der}} \setminus V(I))$.

5.2.2. *Iteration.* In Definition 5.5 we have explained how a localizing sub-category $\mathbf{T}$, containing $\mathbf{S}$, descends to the completion $\mathbf{T}_{\widehat{\mathbf{S}}} \subset \mathsf{C}_{\widehat{\mathbf{S}}}$, and the localization $\mathbf{T}_{(\mathbf{S})} \subset (\mathsf{C}/\mathbf{S})^{\mathrm{ic}}$. We will use this to iterate the completion and localization procedures for a *flag of localizing sub-categories.*

Definition 5.8. *Let C be a stable ∞-category as in Paragraph 5.2.1.*

(a) *A chain of localizing sub-categories $\mathbf{S}_0 \subset \mathbf{S}_1 \subset \dots$, will be referred to as a* flag *in C.*
(b) *We denote $\mathbf{S}_i$ by $\mathsf{C}_{[i]}$.*
(c) *We denote $\mathsf{C}_{\widehat{\mathbf{S}_i}}$ by $\mathsf{C}_{\widehat{[i]}}$.*
(d) *We write $\mathsf{C}_{(i)} = (\mathsf{C}/\mathbf{S}_i)^{\mathrm{ic}}$.*
(e) *We write $\mathsf{C}_{\widehat{(i)}} = (\mathsf{C}_{\widehat{[\mathbf{S}_i]}}/\mathbf{S}_i)^{\mathrm{ic}}$.*
(f) *Given a flag on C as above, we define the* iterated removal-completion operation *by*

$$\mathsf{C}_{\widehat{(0,n)}} = ((\mathsf{C}_{\widehat{(0,n-1)}})_{\widehat{\mathbf{S}_n}}/((\mathbf{S}_n)_{\widehat{(0,n-1)}})^{\mathrm{ic}},$$

with $\mathsf{C}_\emptyset = \mathsf{C}$.

Let X be a scheme. Given a flag of closed subschemes in X,

$$\xi = (X = Z_n \supset Z_{n-1} \supset \cdots \supset Z_0),$$

we obtain a flag of localizing sub-categories $\mathbf{S}_0, \dots, \mathbf{S}_{n-1}$ of $\mathrm{Perf}(X)$ by defining $\mathbf{S}_i$ to be $\mathrm{Perf}_{Z_i}(X)$.

Example 5.9. *Let X be affine n-space $\mathbb{A}^n = \operatorname{Spec} k[t_1, \dots, t_n]$, and ξ the flag given by $Z_i = \operatorname{Spec} k[t_1, \dots, t_i]$. We then have*

$$\mathsf{C}_{\widehat{(0,n)}} \cong \operatorname{Perf}(\operatorname{Spec} k((t_1))\dots((t_n))).$$

This turns out to be a general phenomenon (see also Efimov's [Efi10, Thm. 6.1] for a global analogue), which we state in the following assertion.

Proposition 5.10. *Let X be an excellent reduced scheme. Then the flag of localizing sub-categories $\mathbf{S}_0, \dots, \mathbf{S}_{n-1}$ induced by*

$$\xi \colon X = Z_n \supset Z_{n-1} \supset \cdots \supset Z_0,$$

where we define $\mathbf{S}_i = \operatorname{Perf}_{Z_i}(X)$, satisfies

$$\operatorname{Perf}(X)_{\widehat{(0,n-1)}} \cong \operatorname{Perf}(F_{X,\xi}),$$

where $F_{X,\xi} \cong \mathbb{A}_X(\xi, \mathcal{O}_X)$.

Assume moreover that X is an excellent, reduced k-scheme, where k is a field. For every commutative k-algebra A we have a natural equivalence

$$\operatorname{Perf}(X_A)_{\widehat{(0,n-1)}} \longrightarrow \operatorname{Perf}(A_{X,\xi}),$$

where $A_{X,\xi} \cong \mathbb{A}_{X_A}(\xi, \mathcal{O}_{X_A})$.

The proof will be given in the next paragraph. Reasoning inductively, we will break the lemma down into several steps of independent interest.

5.2.3. *Higher-Dimensional Fields via Categorical Completion.* Recall Proposition 5.2: for R a ring, and an ideal $I \subset R$, the functor $\operatorname{Perf}(R) \longrightarrow \operatorname{Perf}_I(\widehat{R}_I^{\mathrm{der}})$ induces an equivalence

$$\operatorname{Perf}(\widehat{R}_Z^{\mathrm{der}}) \longrightarrow \operatorname{Perf}(R)_{\widehat{\operatorname{Perf}_{V(I)}(R)}}. \tag{23}$$

The following Lemma uses the notion of *equiheighted* ideals, and localization at equiheighted ideals, which were discussed in Definition 4.15.

Lemma 5.11. *Let R be an excellent reduced k-algebra, and $I \subset R$ a radical equiheighted ideal in R, of height 1. Moreover we assume that R is semi-local, i.e. that the set of maximal ideals $\operatorname{Max}(R)$ is finite (therefore defining a closed subset of $\operatorname{Spec} R$). Then, for every k-algebra A, we have a canonical equivalence of stable ∞-categories*

$$\operatorname{Perf}_{V(I_A)}(\operatorname{Spec}((R_A)_I)) \cong \operatorname{Perf}_{V(I_A)}(\operatorname{Spec} R_A \setminus \operatorname{Max}(R)),$$

where $(R_A)_I)$ denotes the ring obtained by localizing R_A at the equiheighted ideal I (see Definition 4.15). In particular, we have for the flag $S_0 = \operatorname{Perf}_{\operatorname{Max}(R)_A}(R_A) \subset \operatorname{Perf}_{V(I_A)}(R_A)$

$$\operatorname{Perf}((R_A)_I)_{[1]} \cong \operatorname{Perf}(R_A)_{(0)[1]},$$

using the notation of Definition 5.8.

Proof. Let $\mathcal{U}^{\mathrm{aff}}$ be the set of affine open subsets $\operatorname{Spec} R_U$ of $\operatorname{Spec} R$, containing all minimal prime ideals above I (i.e. containing the generic points of $V(I) \subset \operatorname{Spec} R$). Inclusion of subsets induces a partial ordering on $\mathcal{U}$. By definition, the localization $(R_A)_I$ can be expressed as the direct limit of rings

$$(R_A)_I \cong \varinjlim_{U \in \mathcal{U}^{\mathrm{aff}}} (R_U)_A.$$

In particular, we obtain

$$\operatorname{Perf}((R_A)_I) \cong \varinjlim_{U \in \mathcal{U}^{\mathrm{aff}}} \operatorname{Perf}((R_U)_A).$$

The same statements are true with support condition, reading as

$$\operatorname{Perf}_{V(I_A)}((R_A)_I) \cong \varinjlim_{U \in \mathcal{U}^{\mathrm{aff}}} \operatorname{Perf}_{V(I_A)}((R_U)_A).$$

Let $\mathcal{U}$ be the set of all open subsets $U \subset \operatorname{Spec} R$, containing all minimal prime ideals above I. Since every open subset is a union of affine open subsets, $\mathcal{U}^{\mathrm{aff}} \subset \mathcal{U}$ is a final directed subset. Hence we have

$$\varinjlim_{U \in \mathcal{U}^{\mathrm{aff}}} \operatorname{Perf}_{V(I_A)}((R_U)_A) \cong \varinjlim_{U \in \mathcal{U}} \operatorname{Perf}_{V(I_A)}(U_A).$$

The following two obervations conclude the proof:

(i) We have $\operatorname{Spec} R \setminus \operatorname{Max}(R) \in \mathcal{U}$.

(ii) All the transition maps in the inverse system computing $\varinjlim_{U \in \mathcal{U}} \operatorname{Perf}_{V(I_A)}(U_A)$ are equivalences. In particular, we have

$$\operatorname{Perf}_{V(I_A)}(U_A) \cong \operatorname{Perf}_{V(I_A)}(\operatorname{Spec}(R_A)_I)$$

for each $U \in \mathcal{U}$.

Assertion (i) follows right from the definition of $\mathcal{U}$: since the minimal prime ideals above I are of height 1, they cannot contain any maximal ideals.

Assertion (ii) fails to hold if one does not impose the support condition. The latter ensures that, for $U \subset V \in \mathcal{U}$ with $U \cap V(I) = V \cap V(I)$, we have that restriction induces an equivalence

$$\operatorname{Perf}_{V(I_A)}(V_A) \longrightarrow \operatorname{Perf}_{V(I_A)}(U_A).$$

Since I has height 1 in R, the open set $V(I) \setminus \operatorname{Max}(R)$ consists precisely of the generic points of $V(I)$. Therefore, every $U \in \mathcal{U}$ intersects $V(I)$ in the same open subset $V(I) \setminus \operatorname{Max}(R)$. As we have just seen this implies that all transition maps

$$\operatorname{Perf}_{V(I_A)}(V_A) \longrightarrow \operatorname{Perf}_{V(I_A)}(U_A)$$

for $U, V \in \mathcal{U}$ are equivalences. The two assertions (i) and (ii) imply now that

$$\operatorname{Perf}_{V(I_A)}(\operatorname{Spec}(R_A)_I) \cong \operatorname{Perf}(\operatorname{Spec} R_A \setminus \operatorname{Max}(R)).$$

The second assertion of the Lemma is merely a reformulation, using the notation introduced in Definition 5.8. $\square$

Corollary 5.12. *Let R and A be a k-algebras, where R is assumed to be Noetherian. We denote by R_A the tensor product $R \otimes_k A$. Let $I_1 \subset I_0 \subset R$ be a chain of equiheighted ideals, such that I_0 induces an ideal of height 1 in R/I_1 (i.e., relative codimension is 1). Using the notation of Definition 5.8, we have a natural equivalence*

$$\operatorname{Perf}_{V(\widehat{I}_1)}(\operatorname{Spec}(\widehat{R}_{A(I_0)})_{\widehat{I}_1}) \longrightarrow \operatorname{Perf}(\operatorname{Spec} R_A)_{\widehat{(I_0)}[I_1]}.$$

Proof. Using Lemma 5.11 we obtain the vertical equivalence in the commutative diagram of stable ∞-categories below

$$\begin{tikzcd}
\operatorname{Perf}(\operatorname{Spec}((\widehat{R_A})_{I_0})_{\widehat{I}_1})_{[I_1]} \arrow[r, "\cong"] \arrow[d, "\cong"'] & \operatorname{Perf}(\operatorname{Spec} R_A)_{\widehat{(I_0)}[I_1]} \\
\operatorname{Perf}(\operatorname{Spec}(\widehat{R_A})_{I_0})_{(I_0)[I_1]}. \arrow[ur] &
\end{tikzcd}$$

According to Definition 5.8, the ∞-category in the bottom left corner agrees with the localization

$$(\operatorname{Perf}_{V(I_1)}(\operatorname{Spec} \widehat{R_A}_{I_0}^{\mathrm{der}}) / \operatorname{Perf}_{V(I_0)}(\operatorname{Spec} \widehat{R_A}_{I_0}^{\mathrm{der}}))^{\mathrm{ic}}.$$

Hence, Proposition 5.2 yields the diagonal functor

$$\operatorname{Perf}(\operatorname{Spec} \widehat{R_A}_{I_0})_{(I_0)[I_1]} \longrightarrow \operatorname{Perf}(\operatorname{Spec} R_A)_{\widehat{(I_0)}[I_1]}.$$

Choosing an inverse for the vertical functor (well-defined up to a contractible space of choices), we obtain the required functor $\operatorname{Perf}_{V(\widehat{I}_1)}(\operatorname{Spec}(\widehat{R_A}_{I_0})_{\widehat{I}_1}) \longrightarrow \operatorname{Perf}(\operatorname{Spec} R_A)_{\widehat{(I_0)}[I_1]}$. $\square$

Proof of Proposition 5.10. We only give the proof of the second assertion, i.e. for X a scheme over k. The first assertion is proven analogously. We may assume without loss of generality that X is affine, since Z_0 is a finite union of closed points. Thus, let R be a k-algebra, such that $X \cong \operatorname{Spec} R$.

Recall from Lemma 4.18 that $A_{X,\xi}$ can be obtained by iteratively completing and localizing R at a chain of equiheighted ideals $I_0 \supset I_1 \supset \cdots \supset I_{n-1}$, corresponding to the closed subschemes $Z_0 \subset \cdots \subset Z_{n-1}$. We will use analogous notation for the ring

$$A_{X,\xi} = (\mathsf{L} \circ \mathsf{C})^n R_A.$$

The asserted equivalence is a special case of the more general statement

$$\mathrm{Perf}_{(Z_k)_A}((\mathsf{L} \circ \mathsf{C})^k R_A) \longrightarrow \mathrm{Perf}(X_A)_{\widehat{(0,k-1)}[k]}, \tag{24}$$

which will be proven inductively.

Equation (24) for $k = 0$ amounts to the definition of $\mathbf{S}_0$:

$$\mathrm{Perf}_{(Z_0)_A}(R_A) \cong \mathrm{Perf}(X_A)_{[0]} = \mathbf{S}_0.$$

This will be the anchor point of our induction.

We will prove that equation (24) holds for $k = m+1$ if it holds for $k = m$. Taking Ind-objects of both sides, and considering the (idempotent completion of the) essential image of $\mathrm{Perf}_{(Z_{m+1})_A}(R_A) \cong \mathrm{Perf}_{(Z_A)_{m+1}}(X_A)$, we see that (up to idempotent completion)

$$\mathrm{Perf}((\mathsf{L} \circ \mathsf{C})^m R_A)_{\widehat{[m]}[m+1]} \cong \mathrm{Im}[\mathrm{Perf}_{(Z_{m+1})_A}(R_A) \longrightarrow \mathsf{Ind}\,\mathrm{Perf}_{(Z_m)_A}((\mathsf{L} \circ \mathsf{C})^m R_A)],$$

here we use that for a stable ∞-category C endowed with a flag of localizing sub-categories, the functor $\mathsf{C}_{[i]} \longrightarrow \mathsf{C}_{\widehat{(0,i)}[i+1]}$ is essentially surjective up to idempotent completion. We have

$$\mathrm{Im}[\mathrm{Perf}(X_A)_{[m+1]} \longrightarrow \mathsf{Ind}\,\mathrm{Perf}(X_A)_{\widehat{(0,m-1)}[m]})] \cong \mathrm{Perf}(X_A)_{\widehat{(0,m-1)}\widehat{[m]}[m+1]},$$

by virtue of Definition 5.1.

Since we are completing perfect complexes on the affine scheme $\mathrm{Spec}(\mathsf{L} \circ \mathsf{C})^m R_A)$, Proposition 5.2 gives rise to a canonical functor

$$\mathrm{Perf}(\mathsf{C}(\mathsf{L} \circ \mathsf{C})^m R_A)_{[m+1]} \longrightarrow \mathrm{Perf}((\mathsf{L} \circ \mathsf{C})^m R_A)_{\widehat{[m]}[m+1]} \cong \mathrm{Perf}(X_A)_{\widehat{(0,m)}[m+1]},$$

which is an equivalence.

Corollary 5.12 yields an equivalence

$$\mathrm{Perf}(\mathsf{L} \circ \mathsf{C}(\mathsf{L} \circ \mathsf{C})^m R_A)_{[m+1]} \cong \mathrm{Perf}(\mathsf{C}(\mathsf{L} \circ \mathsf{C})^m R_A)_{(m)[m+1]}.$$

Pairing this with the functoriality of localizing at the m-th localizing sub-category, we therefore obtain an equivalence

$$\mathrm{Perf}((\mathsf{L} \circ \mathsf{C})^{m+1} R_A)_{[m+1]} \longrightarrow \mathrm{Perf}(R_A)_{\widehat{(0,m)}[m+1]},$$

of stable ∞-categories. $\square$

By similar techniques one proves the following:

Theorem 5.13. *Let X be an excellent, reduced k-scheme of pure dimension n, where k is a field, and A a k-algebra.*

(a) *We denote by $\mathbf{S}_j \subset \mathrm{Perf}(X_A)$ the localizing sub-category given by the union of the sub-categories $\mathrm{Perf}_{Z_A}(X_A)$ with $\dim Z \leq j$. Then we have*

$$\mathrm{Perf}(\mathbb{A}_X(|X|_n^{\mathrm{red}}, \mathcal{O}_X)\widehat{\otimes}_k A) \cong \mathrm{Perf}(X_A)_{\widehat{(0,n)}}.$$

(b) *Let $\xi\colon X = Z_n \supset \cdots \supset Z_{i+1} \supset Z_{i-1} \supset \cdots \supset Z_0$ be an* almost saturated flag *of equiheighted closed subschemes, satisfying $\dim Z_j = j$. Let $T_\xi \subset |X|_n^{\mathrm{red}}$ be the subset of reduced chains $\eta_0 < \eta_1 < \cdots \eta_n$, such that for $j \neq i$ we have that η_j is a generic point of Z_j. Then we have the equivalence $\mathrm{Perf}(\mathbb{A}_X(T_\xi, \mathcal{O}_X)\widehat{\otimes}_k A) \cong \mathrm{Perf}(X_A)_{\widehat{(0,n)}}$.*

6. Reciprocity

Let X be a proper, reduced curve over a field k. For every commutative k-algebra A, and a pair of units in the ring of *A-valued rational functions*

$$f, g \in A(X)^{\times} = (k(X) \otimes_k A)^{\times}$$

we obtain an element of $A^{\times}$ given by $(f, g)_x$ for every closed point $x \in X$.

Theorem 6.1 (Weil, Anderson–Pablos Romo, Beilinson–Bloch–Esnault)**.** *The product below is well-defined and satisfies*

$$\prod_{x \in X_0} (f, g)_x = 1.$$

This reciprocity law has been proven by Weil for $A = k$, it was generalized to the case of artinian rings A by Anderson–Pablos Romo [APR04], and to general A by Beilinson–Bloch–Esnault [BBE02, Section 3.4]. Recently, Pál has shown in [Pál10] that, for artinian rings, the relative version of Weil reciprocity follows from the absolute case ($A = k$) after a change of fields.

This section is concerned with an extension of this result to varieties of arbitrary dimension (and arbitrary rings A). The absolute case ($A = k$) is due to Kato [Kat86] (however, the case of surfaces was pioneered by Parshin). Recent work of Osipov–Zhu [OZ13] established a Contou-Carrère reciprocity law for surfaces and artinian rings.

As before, A denotes a k-algebra over a field k. The main player is an n-dimensional, separated k-scheme of finite type X, *together* with an *almost saturated flag*

$$\zeta = (X = Z_n \supset \cdots Z_{i+1} \supset Z_{i-1} \supset \cdots Z_0), \tag{25}$$

of closed irreducible subvarieties, satisfying $\dim Z_j = j$. For every closed equiheighted i-dimensional subset Z, satisfying $Z_{i+1} \supset Z \supset Z_{i-1}$ we obtain a *saturated flag* ξ_Z. Note that we denote saturated flags by the letter ξ for the sake of visual distinctness.

In order to formulate the reciprocity law, we need to construct an analogue of the ring of A-valued rational functions $A(X)$ on a curve X. This ring $A_\zeta(X)$ should be naturally associated to the data (X, ζ) and the k-algebra A. Further, for each Z as above, we require a specialization homomorphism

$$A_\zeta(X) \longrightarrow A_{X, \xi_Z}.$$

The latter is required to make sense of the factors of the product

$$\prod_{Z_{i+1} \supset Z \supset Z_{i-1}} (f_0, \ldots, f_n)_{\xi_Z}.$$

As a first step towards a complete statement, we define the analogue of the rings $A(X)$ above. The definition below is easier to understand once one realizes that $A(X)$ can expressed as the direct limit $A(X) \cong \varinjlim \Gamma(X \setminus Z, \mathcal{O}) \otimes_k A$, where Z ranges over the collection of closed 0-dimensional subsets.

Definition 6.2. *Let X be a separable n-dimensional k-scheme of finite type, A a k-algebra, and ζ an almost complete flag in X. For each $Z_{i+1} \supset Z \supset Z_{i-1}$ with Z of pure dimension i and not necessarily irreducible, we denote the ring of regular functions on the scheme $(\mathsf{C} \circ \mathsf{L})^{n-i-1} \circ \mathsf{L} \circ (\mathsf{C} \circ \mathsf{L})^i (X_A, (\xi_Z)_A)$ by $A_{\zeta,Z}(X)$. We define the ring $A_\zeta(X)$ to be the direct limit*

$$A_\zeta(X) = \varinjlim_{Z_{i-1} \subset Z \subset Z_{i+1}} A_{\zeta,Z}(X),$$

where Z is a closed subset of pure dimension i (not necessarily irreducible).

After having introduced this colimit, we observe that the algebraic K-theory is manageable for formal reasons. This will be used in the proof of our main result.

Remark 6.3. *Since non-connective K-theory of rings commutes with filtered colimits (Theorem 7.2 of* [TT90]*) one has*

$$\mathbb{K}_{A_{X,\zeta}} = \varinjlim_{Z_{i-1} \subset Z \subset Z_{i+1}} \mathbb{K}_{(\mathsf{C} \circ \mathsf{L})^{n-i-1} \circ \mathsf{L} \circ (\mathsf{C} \circ \mathsf{L})^i (X_A, (\xi_Z)_A)}.$$

We are now ready to state the main result of this section, in a classical formulation:

Theorem 6.4 (Reciprocity for Contou-Carrère symbols)**.** *Let X be a reduced, n-dimensional k-scheme of finite type, and let A be a commutative k-algebra. Let ζ be an almost saturated flag as in* (25)*. For every $(n+1)$-tuple $f_0, \ldots, f_n \in A_\zeta(X)^\times$ we have that the product below is well-defined and satisfies the identity*

$$\prod_{Z_{i+1} \supset Z \supset Z_{i-1}} (f_0, \ldots, f_n)_{\xi_Z} = 1,$$

where Z is irreducible and of dimension i.

We will deduce this result in Subsection 6.2 from an *abstract reciprocity law* for compositions of boundary maps (see Corollary 6.9). The reciprocity relation will be generalized to the existence of a null-homotopy for a certain map of spectra. We refer to such a construction as *spectrification* (following Beilinson).

6.1. **Abstract Reciprocity Laws.** In the following we denote by C a stable ∞-category, and consider a chain $\mathbf{S}_0, \mathbf{S}_1, \ldots, \mathbf{S}_n$ of localizing subcategories (as considered in Subsection 5.2). We will be concerned with the composition of boundary maps, connecting the non-connective K-theory spectra of various stable ∞-categories constructed from C with the help of the localizing subcategories.

Theorem 6.5 (Abstract Weil reciprocity)**.** *Let C be a stable ∞-category together with a localizing subcategory $\mathbf{S}$. We assume that there exists a commutative diagram of functors*

$$\begin{tikzcd} \mathbf{S} \arrow[r] \arrow[dr] & \mathsf{C} \arrow[d] \\ & \mathsf{D}, \end{tikzcd}$$

where D denotes as well a stable ∞-category. Under these assumptions the map $\Omega\,\mathbb{K}_{\mathsf{C}_{(\mathbf{S})}} \longrightarrow \mathbb{K}_{\mathsf{D}}$ defined by

$$\begin{tikzcd} \Omega\,\mathbb{K}_{\mathsf{C}_{(\mathbf{S})}} \arrow[r] \arrow[d] & \mathbb{K}_{\mathsf{D}} \\ \Omega\,\mathbb{K}_{\mathsf{C}_{\widehat{(\mathbf{S})}}} \arrow[r] & \mathbb{K}_{\mathbf{S}} \arrow[u] \end{tikzcd}$$

is homotopic to the zero map.

Proof. Commutativity of the diagram above follows from the naturality of boundary maps in algebraic K-theory. We may therefore focus on establishing the null-homotopy of the map $\Omega\,\mathbb{K}_{\mathsf{C}_{(\mathbf{S})}} \longrightarrow \mathbb{K}_{\mathsf{D}}$.

We have a commuting diagram of spectra, with the square being bicartesian:

$$\begin{tikzcd} & \Omega\,\mathbb{K}_{\mathsf{C}_{(\mathbf{S})}} \arrow[r] \arrow[d] \arrow[ddl] & 0 \arrow[d] \\ & \mathbb{K}_{\mathbf{S}} \arrow[r] \arrow[dl] & \mathbb{K}_{\mathsf{C}} \arrow[dll] \\ \mathbb{K}_{\mathsf{D}} & & \end{tikzcd}$$

□

Example 6.6 (Weil reciprocity)**.** *Let X be a proper, reduced curve over a field k, we set $\mathsf{C} =$ $\mathrm{Perf}(X)$, and for every 0-dimensional closed subscheme Z (not assumed to be irreducible) we let $\mathbf{S}$ be $\mathrm{Perf}_{|Z|}(X)$. We then have $\mathsf{C}_{(\mathbf{S})} \cong \mathrm{Perf}(X \setminus Z)$. Using the pushforward functor to the base field $\pi_*\colon \mathrm{Perf}(X) \longrightarrow \mathrm{Perf}(k)$, Theorem 6.5 implies now that the canonical map*

$$\Omega\,\mathbb{K}_{X \setminus Z} \longrightarrow \mathbb{K}_k$$

is homotopic to zero.

The field of rational functions $k(X)$ arises as the inverse limit

$$k(X) \cong \varinjlim_{Z} \mathcal{O}_{X \setminus Z},$$

in particular we have $\mathbb{K}_{k(X)} \cong \varinjlim_Z \mathbb{K}_{X\setminus Z}$, by virtue of Theorem 7.2 in [TT90].

Since we may identify the direct limit of the ∞-categories $\mathrm{Perf}(X)_{\widehat{(\mathbf{S})}}$ with $\mathrm{Perf}(\mathbb{A}_X)$, by virtue of Theorem 5.13(a), we obtain a commutative diagram

$$\begin{tikzcd}
\Omega\,\mathbb{K}_{k(X)} \arrow[r] \arrow[d] & \mathbb{K}_k \\
\Omega\,\mathbb{K}_{\mathbb{A}_X}\,. \arrow[ur] &
\end{tikzcd}$$

Passing to homotopy groups we obtain a commutative diagram of abelian groups

$$\begin{tikzcd}
K_2(k(X)) \arrow[r] \arrow[d] & K_1(k) \\
K_2(\mathbb{A}_X), \arrow[ur] &
\end{tikzcd}$$

thus we see that $\prod_{x\in X_0} \pi_*\partial\{f,g\} = \prod_{x\in X_0} N_{\kappa(x)/k}(f,g)_x = 1$, for all pairs of invertible rational functions on X.

Similarly one could use this result to prove reciprocity for Contou-Carrère symbols, relative to any k-algebra A. We will give more details at the end of this section, when discussing the proof of reciprocity for higher-dimensional varieties.

Theorem 6.7 (Abstract Parshin reciprocity)**.** *We denote by C a stable ∞-category, and by $\mathbf{S}_0 \subset \mathbf{S}_1 \subset \mathsf{C}$ a length 2 chain of localizing subcategories. Then the following commuting diagram*

$$\begin{tikzcd}
\Omega^2\,\mathbb{K}_{\mathsf{C}_{\widehat{(0)}(1)}} \arrow[r, "\partial"] \arrow[d] & \Omega\,\mathbb{K}_{\mathsf{C}_{\widehat{(0)}[1]}} \arrow[r, "\partial"] \arrow[d, "\cong"] & \mathbb{K}_{\mathsf{C}_{[0]}} \\
\Omega^2\,\mathbb{K}_{\mathsf{C}_{\widehat{(0,1)}}} \arrow[r, "\partial"] & \Omega\,\mathbb{K}_{\mathsf{C}_{\widehat{(0)}[1]}} \arrow[ur, "\partial"] &
\end{tikzcd}$$

exists, and the composition of the top row is equivalent to the zero map.

Proof. As in the proof of Abstract Weil reciprocity, the existence of the commutative diagram follows directly from the naturality of boundary maps. We therefore turn to proving the existence of a null-homotopy for the composition of the top row. Similar to the proof of Theorem 6.5 we show that this composition factors through the juxtaposition of two subsequent maps in an exact sequence of spectra (thus is homotopic to 0). This is achieved by the commuting diagram

$$\begin{tikzcd}
 & \Omega^2\,\mathbb{K}_{\mathsf{C}_{\widehat{(0)}(1)}} \arrow[r] \arrow[d] \arrow[ddl] & 0 \arrow[d] \\
 & \Omega\,\mathbb{K}_{\widehat{(0)}[1]} \arrow[r] \arrow[dl] & \Omega\,\mathbb{K}_{\mathsf{C}_{\widehat{(0)}}}, \arrow[dll, dashed] \\
\mathbb{K}_{\mathsf{C}_{[0]}} & &
\end{tikzcd}$$

provided we can establish the existence of the dashed map. A suitable candidate for this map is given by the K-theory boundary morphism of the exact sequence of stable ∞-categories

$$\mathsf{C}_{[0]} \hookrightarrow \mathsf{C}_{\widehat{[0]}} \twoheadrightarrow \mathsf{C}_{\widehat{(0)}}.$$

Naturality of boundary maps implies the existence of a commutative diagram with exact rows

$$\begin{tikzcd}
& 0 \arrow[rr] \arrow[dd] & & \mathbb{K}_{\mathsf{C}_{\widehat{[0]}[1]}} \arrow[dd] \\
\Omega\,\mathbb{K}_{\mathsf{C}_{\widehat{(0)}[1]}} \arrow[ur] \arrow[rr] \arrow[dd] & & \mathbb{K}_{\mathsf{C}_{[0]}} \arrow[ur] \arrow[dd] & \\
& 0 \arrow[rr] & & \mathbb{K}_{\mathsf{C}_{\widehat{[0]}}} \\
\Omega\,\mathbb{K}_{\mathsf{C}_{\widehat{(0)}}} \arrow[ur] \arrow[rr, dashed] & & \mathbb{K}_{\mathsf{C}_{[0]}} \arrow[ur] & .
\end{tikzcd}$$

The square in front amounts to the existence of the commuting triangle containing the dashed map above. This concludes the proof. ☐

Let us explain how this result implies Parshin's reciprocity statement for surfaces.

Example 6.8 (Parshin reciprocity)**.** *Let Y be a separated, excellent, reduced surface. We denote by C the stable ∞-category $\mathrm{Perf}(Y)$. For a fixed closed point $x \in Y$, we obtain a localizing subcategory $\mathbf{S}_0 = \mathrm{Perf}_{\{x\}}(Y)$. Moreover, for every curve C, with $x \in C$, we have a localizing subcategory $\mathbf{S}_1 = \mathrm{Perf}_{|C|}(Y)$.*

Theorem 5.13(a) implies that $\mathsf{C}_{\widehat{(0,1)}} \cong \mathrm{Perf}(\mathbb{A}_{Y,C,x})$, and a direct limit of the ∞-categories $\mathsf{C}_{\widehat{(0)}(1)}$ yields $\mathrm{Perf}(\mathrm{Frac}(\widehat{\mathcal{O}_{Y,x}}))$. Hence, by Theorem 6.7, we have a commutative diagram of K-theory groups

$$\begin{tikzcd}
K_3(\mathrm{Frac}(\widehat{\mathcal{O}_{Y,x}})) \arrow[r] \arrow[d] & K_1(\mathcal{O}_{Y,x}/\mathfrak{m}_{Y,x}) \\
K_3(\mathbb{A}_{Y,?,x}), \arrow[ur] &
\end{tikzcd}$$

in which the top map is trivial. Here $\mathbb{A}_{Y,?,x}$ denotes the ring of adèles for chains, $Y \supset C \supset \{x\}$, where C can be an arbitrary irreducible curve containing x. In particular, we see that for every triple $f_0, f_1, f_2 \in \mathrm{Frac}(\widehat{\mathcal{O}_{Y,x}})^\times$ we have the identity

$$\prod_{C \ni x} (f_0, f_1, f_2)_{x \in C} = 1,$$

where the product is indexed by irreducible curves containing x.

Combining Theorems 6.5 and 6.7, we obtain an abstract analogue of Kato reciprocity. In the next subsection we will use this result to deduce a reciprocity law for Contou-Carrère symbols.

Corollary 6.9 (Abstract Kato reciprocity)**.** *Let C be a stable ∞-category. We fix positive integers i and n, and assume that we have a chain of localizing subcategories $\mathbf{S}_j \subset \mathsf{C}$, indexed by $0 \leq j \leq n-1$.*

(a) *If $i = 0$, suppose that we have a commutative diagram*

$$\begin{tikzcd}
\mathbf{S}_0 \arrow[r] \arrow[dr] & \mathbf{S}_1 \arrow[d] \\
& \mathsf{D},
\end{tikzcd}$$

where D *denotes a stable* ∞*-category. Then, the following commuting diagram*

$$\begin{tikzcd} \Omega^n \mathbb{K}_{\mathsf{C}_{(0)(\widehat{1,n-1})}} \arrow[r] \arrow[d] & \mathbb{K}_{\mathsf{D}} \\ \Omega^n \mathbb{K}_{\mathsf{C}_{(\widehat{0,n-1})}} \arrow[r] & \mathbb{K}_{\mathsf{C}_{[0]}} \arrow[u] \end{tikzcd}$$

exists, and the top map is homotopic to zero.

(b) *If* $i \neq 0$*, then the following commuting diagram*

$$\begin{tikzcd} \Omega^n \mathbb{K}_{\mathsf{C}_{(\widehat{0,i-1})(i)(\widehat{i+1,n-1})}} \arrow[r] \arrow[d] & \mathbb{K}_{\mathsf{C}_{[0]}} \\ \Omega^n \mathbb{K}_{\mathsf{C}_{(\widehat{0,n-1})}} \arrow[ur, "\partial^n"'] & \end{tikzcd}$$

exists, and the top row is null-homotopic.

Proof. The first assertion follows directly from Theorem 6.5, when setting $\mathsf{C} = \mathbf{S}_1$, and $\mathbf{S} = \mathbf{S}_0$.

We will now turn to the proof of the second assertion. For $j \leq i-1$ we denote by

$$\partial_j \colon \Omega^{j+1} \mathbb{K}_{\mathsf{C}_{(\widehat{0,j})[j+1]}} \longrightarrow \Omega^j \mathbb{K}_{\mathsf{C}_{(\widehat{0,j-1})[j]}} \tag{26}$$

the boundary morphism in K-theory, associated to the short exact sequence

$$\mathsf{C}_{(\widehat{0,j-1})[j]} \hookrightarrow \mathsf{C}_{(\widehat{0,j-1})\widehat{[j]}[j+1]} \twoheadrightarrow \mathsf{C}_{(\widehat{0,j})[j+1]} \tag{27}$$

of stable ∞-categories. Analogously, we have the boundary maps

$$\partial_i \colon \Omega^{i+1} \mathbb{K}_{\mathsf{C}_{(\widehat{0,i-1})(i)[i+1]}} \longrightarrow \Omega^i \mathbb{K}_{\mathsf{C}_{(\widehat{0,i-1})[i]}}, \tag{28}$$

and for $j \geq i+1$

$$\partial_j \colon \Omega^{j+1} \mathbb{K}_{\mathsf{C}_{(\widehat{0,i-1})(i)(\widehat{i+1,j})[j+1]}} \longrightarrow \Omega^j \mathbb{K}_{\mathsf{C}_{(\widehat{0,j-1})(i)(\widehat{i+1,j-1})[j]}}. \tag{29}$$

We want to show that the composition of these boundary maps satisfies

$$\partial_0 \circ \cdots \circ \partial_{n-1} \simeq 0.$$

In fact, Theorem 6.7 implies that $\partial_{i-1} \circ \partial_i \simeq 0$. To see this one chooses the C in *loc. cit.* to be the stable ∞-category $\mathsf{C}_{(\widehat{0,i-2})[i+1]}$, $\mathbf{S}_0 = \mathsf{C}_{(\widehat{0,i-2})[i-1]}$, and $\mathbf{S}_1 = \mathsf{C}_{(\widehat{0,i-2})[i]}$. ☐

Example 6.10 (Kato reciprocity)**.** *Let* X *be a separated, reduced, excellent scheme of pure dimension* n*. Let* ζ *denote an almost saturated flag of closed irreducible subschemes*

$$\zeta = (X \supset Z_{n-1} \supset \cdots \supset Z_{i+1} \supset Z_{i-1} \supset \cdots \supset Z_0),$$

indexed by $j \neq i$*, with* $\dim Z_j = j$*. For every (not necessarily irreducible) closed subscheme* Z_i *of pure dimension* i*, and* $Z_{i+1} \supset Z \supset Z_{i-1}$ *we obtain a natural chain of localizing subcategories* $\mathbf{S}_j := \mathrm{Perf}_{|Z_j|}$ *on* $\mathsf{C} = \mathrm{Perf}(X)$*. If* $i = 0$*, we assume that* Z_1 *is proper over a field* k*. Abstract Kato reciprocity now implies the existence of a commutative diagram*

$$\begin{tikzcd} \Omega^n \mathbb{K}_{F_{X,\zeta}} \arrow[r, "0"] \arrow[d] & \mathbb{K}_{\mathsf{D}}, \\ \Omega^n \mathbb{K}_{\mathbb{A}_{X,\zeta}}, \arrow[ur, "\partial^n"'] & \end{tikzcd}$$

where we let $\mathsf{D} = \mathrm{Perf}(k)$ *for* $i = 0$*, and* $\mathrm{Perf}_{Z_0}(X)$ *otherwise. As before, this implies that for an* $(n+1)$*-tuple of invertible elements of* $F_{X,\zeta}$*, we have*

$$\prod_{Z_{i+1} \supset Z \supset Z_{i-1}} (f_0, \ldots, f_n)_{\xi_Z} = 1,$$

where Z is irreducible and of dimension i.

6.2. **Reciprocity for Contou-Carrère Symbols.** In the following we fix a separable, reduced k-scheme X of finite type and dimension n, a k-algebra A, and an integer i. As in Example 6.10, ζ denotes an almost saturated flag of closed irreducible subschemes

$$\zeta = (X \supset Z_{n-1} \supset \cdots \supset Z_{i+1} \supset Z_{i-1} \supset \cdots \supset Z_0),$$

indexed by $0 \leq j \leq n$ with $j \neq i$, and satisfying $\dim Z_j = j$. The condition of being *almost saturated* stipulates that up to the choice of $Z_{i+1} \supset Z_i \supset Z_{i-1}$, the flag cannot be further extended. If $i = 0$, we assume that Z_1 is proper over a field k.

Alluding to the notation of abstract Kato reciprocity (Corollary 6.9), we define

$$\mathsf{C} = \mathrm{Perf}(X_A),\ \mathbf{S}_j = \mathrm{Perf}_{|(Z_j)_A|}(X_A),$$

where $Z_i = Z$ is a not necessarily irreducible closed subset of pure dimension i, satisfying $Z_{i+1} \supset Z \supset Z_{i-1}$.

Lemma 6.11. *Using the notation introduced earlier, we have the following equivalences.*

(a) $\mathsf{C}_{\widehat{(0,i-1)}(i)\widehat{(i+1,n-1)}} \cong \mathrm{Perf}((\mathsf{C}\circ\mathsf{L})^{n-i-1}\circ\mathsf{L}\circ(\mathsf{C}\circ\mathsf{L})^{i}(X_A,(\xi_Z)_A))$ *(see Definition 6.2). In particular, taking the colimit of the diagram of these stable ∞-categories indexed by all possible Z, we obtain* $\mathrm{Perf}(A_\zeta(X))$.

(b) *For each $Z_{i+1} \supset Z \supset Z_{i-1}$,* $\mathsf{C}_{\widehat{(0,n-1)}} \cong \mathrm{Perf}(\mathbb{A}_{X,\xi_Z,A})$

Proof. The second assertion is a direct consequence of Theorem 5.13(c). The first assertion is proven by similar means as the results in Subsection 5.2.3: as in *loc. cit.* one proceeds by induction, where the i-th step (due to the absence of completion) has to be treated separately (using Lemma 5.11 instead of Corollary 5.12). □

Using the equivalences of stable ∞-categories, provided by Lemma 6.11, abstract Kato reciprocity implies the following corollary.

Corollary 6.12 (Spectral Contou-Carrère reciprocity)**.** *The following diagram of spectra commutes*

$$\begin{tikzcd} \Omega^n\,\mathbb{K}_{A_{\zeta,Z}(X)} \arrow[r] \arrow[d] & \mathbb{K}_A\,. \\ \Omega^n\,\mathbb{K}_{\mathbb{A}_{X,\xi_Z,A}}, \arrow[ur, "\sigma^A_{X,\xi_Z}"'] & \end{tikzcd}$$

Taking the colimit over Z, we obtain a commuting triangle of spectra

$$\begin{tikzcd} \varinjlim_Z \Omega^n\,\mathbb{K}_{A_{\zeta,Z}(X)} \arrow[r] \arrow[d] & \mathbb{K}_A \\ \varinjlim_Z \Omega^n\,\mathbb{K}_{A_{X,\xi_Z}}\,. \arrow[ur, "\varinjlim \sigma^A_{X,\xi_Z}"'] & \end{tikzcd}$$

Moreover, the top row in both diagrams is null-homotopic.

Proof. Using Remark 6.3 one obtains the second commuting triangle from the first (including the null-homotopy), by taking a colimit ranging over the collection of all possible $Z_{i+1} \supset Z \supset Z_{i-1}$. At the beginning of this subsection, we have already defined a chain of localizing subcategories $\mathbf{S}_j$ on $\mathsf{C} = \mathrm{Perf}(X_A)$, which allows us to evoke abstract Kato reciprocity (Corollary 6.9). We only need to verify that one of the conditions (a) or (b) holds, in order to apply this result. If $i = 0$, then Z_1 is proper over k by assumption. By virtue of Lemma 4.37 we obtain a pushforward functor

$$\pi_*\colon \mathbf{S}_1 \cong \mathrm{Perf}_{(Z_1)_A}(X_A) \longrightarrow \mathrm{Perf}(A),$$

which yields the required commutative diagram

$$\begin{tikzcd} \mathrm{Perf}_{(Z_0)_A}(X_A) \arrow[r] \arrow[dr] & \mathrm{Perf}_{(Z_1)_A}(X_A) \arrow[d] \\ & \mathrm{Perf}(A). \end{tikzcd}$$

If $i \geq 1$, there is nothing to check. □

Proof of Theorem 6.4. Let $f_0, \ldots, f_n$ be a commuting $(n+1)$-tuple of units in the ring $A_\zeta(X)$. This corresponds to a map $\Sigma^\infty \mathbb{T}^{n+1} \longrightarrow \Sigma^\infty(BA_\zeta(X)^\times)$. The right hand side can be expressed as a colimit by definition of the ring $A_\zeta(X)$ (see Definition 6.2). Because the torus is compact, the map factors through a map $\Sigma^\infty \mathbb{T}^{n+1} \longrightarrow \Sigma^\infty(BA_{\zeta,Z}(X)^\times)$ for some Z.

The ring $A_{\zeta,Z}(X)$ splits into a product over the irreducible components of $Z = \bigcup_{k=1}^m W_k$. Therefore, spectral Contou-Carrère reciprocity 6.12 yields a commutative diagram

$$\begin{tikzcd} \bigoplus_{k=1}^m \Omega^n \mathbb{K}_{A_{\zeta,W_k}(X)} \arrow[r, "\cong"] & \Omega^n \mathbb{K}_{A_{\zeta,Z}(X)} \arrow[r, "0"] \arrow[d] & \mathbb{K}_A \\ & \Omega^n \mathbb{K}_{\mathbb{A}_{X,\xi_Z,A}} . \arrow[ur, "\sigma^A_{X,\xi_Z}"'] & \end{tikzcd}$$

Passing to homotopy groups, and applying the resulting maps to the object represented by the Steinberg symbol $\{f_0, \ldots, f_n\}$ (i.e. a higher commutator by Proposition 3.31), we obtain the identity

$$\prod_{i=1}^m \pi_* \partial^n_{W_i}\{f_0, \ldots, f_n\} = \prod_{i=1}^m (f_0, \ldots, f_n)_{\xi_{W_i}} = 1.$$

This concludes the proof. □

References

[ADCK89] E. Arbarello, C. De Concini, and V. G. Kac, *The infinite wedge representation and the reciprocity law for algebraic curves*, Theta functions—Bowdoin 1987, Part 1 (Brunswick, ME, 1987), Proc. Sympos. Pure Math., vol. 49, Amer. Math. Soc., Providence, RI, 1989, pp. 171–190. MR 1013132 (90i:22034)

[APR04] G.W. Anderson and F. Pablos Romo, *Simple proofs of classical explicit reciprocity laws on curves using determinant groupoids over an Artinian local ring*, Comm. Algebra **32** (2004), no. 1, 79–102. MR 2036223 (2005d:11099)

[ATJLL97] Leovigildo Alonso Tarrío, Ana Jeremías López, and Joseph Lipman, *Local homology and cohomology on schemes*, Ann. Sci. École Norm. Sup. (4) **30** (1997), no. 1, 1–39. MR 1422312 (98d:14028)

[BBE02] A. Beilinson, S. Bloch, and H. Esnault, *ϵ-factors for Gauss-Manin determinants*, Mosc. Math. J. **2** (2002), no. 3, 477–532, Dedicated to Yuri I. Manin on the occasion of his 65th birthday. MR 1988970 (2004m:14011)

[Bei80] A. Beilinson, *Residues and adèles*, Funktsional. Anal. i Prilozhen. **14** (1980), no. 1, 44–45. MR 565095 (81f:14010)

[Bei87] ———, *How to glue perverse sheaves*, K-theory, arithmetic and geometry (Moscow, 1984–1986), Lecture Notes in Math., vol. 1289, Springer, Berlin, 1987, pp. 42–51. MR 923134 (89b:14028)

[BGT13] Andrew J. Blumberg, David Gepner, and Gonçalo Tabuada, *A universal characterization of higher algebraic K-theory*, Geom. Topol. **17** (2013), no. 2, 733–838. MR 3070515

[BGW14a] O. Braunling, M. Groechenig, and J. Wolfson, *The index map in algebraic K-theory*, arXiv:1410.1466, 10 2014.

[BGW14b] ———, *Tate Objects in Exact Categories (with appendix by Jan Stovicek and Jan Trlifaj)*, arXiv:1402.4969, 02 2014.

[BM96a] J.-L. Brylinski and D. A. McLaughlin, *Characteristic classes and multidimensional reciprocity laws*, Math. Res. Lett. **3** (1996), no. 1, 19–30. MR 1393379 (98g:19001)

[BM96b] ———, *Multidimensional reciprocity laws*, J. Reine Angew. Math. **481** (1996), 125–147. MR 1421948 (98b:11072)

[CC94] C. Contou-Carrère, *Jacobienne locale, groupe de bivecteurs de Witt universel, et symbole modéré*, C. R. Acad. Sci. Paris Sér. I Math. **318** (1994), no. 8, 743–746. MR 1272340 (95c:14059)

[Coh46] I. S. Cohen, *On the structure and ideal theory of complete local rings*, Trans. Amer. Math. Soc. **59** (1946), 54–106. MR 0016094 (7,509h)

[Con07] Brian Conrad, *Deligne's notes on Nagata compactifications*, J. Ramanujan Math. Soc. **22** (2007), no. 3, 205–257. MR 2356346 (2009d:14002)

[CP99] R. Carey and J. Pincus, *Joint torsion of Toeplitz operators with h^∞ symbols*, Integral Equations Operator Theory **33** (1999), no. 3, 273–304.

[Del91] P. Deligne, *Le symbole modéré*, Inst. Hautes Études Sci. Publ. Math. (1991), no. 73, 147–181. MR 1114212 (93i:14030)

[DG02] W.G. Dwyer and J.P.C. Greenlees, *Complete modules and torsion modules*, Amer. J. Math. **124** (2002), no. 1, 199–220.

[Dri06] Vladimir Drinfeld, *Infinite-dimensional vector bundles in algebraic geometry: an introduction*, The unity of mathematics, Progr. Math., vol. 244, Birkhäuser Boston, Boston, MA, 2006, pp. 263–304. MR 2181808 (2007d:14038)

[Efi10] A.I. Efimov, *Formal completion of a category along a subcategory*, arXiv:1006.4721, 06 2010.

[FV02] I. B. Fesenko and S. V. Vostokov, *Local fields and their extensions*, second ed., Translations of Mathematical Monographs, vol. 121, American Mathematical Society, Providence, RI, 2002, With a foreword by I. R. Shafarevich. MR 1915966 (2003c:11150)

[GM92] J.P.C. Greenlees and J.P. May, *Derived functors of I-adic completion and local homology*, J. Algebra **149** (1992), no. 2, 438–453.

[GR11] Dennis Gaitsgory and Nick Rozenblyum, *DG Indschemes*, arXiv:1108.1738, 08 2011.

[Gro10] Moritz Groth, *A short course on infinity-categories*, arXiv:1007.2925, 07 2010.

[HH75] W.J. Helton and R.E. Howe, *Traces of commutators of integral operators*, Acta Math. **135** (1975), no. 3-4, 271–305.

[HL13] I. Horozov and Z. Luo, *On the Contou-Carrère Symbol for Surfaces*, arXiv:1310.7065, October 2013.

[Hub91] A. Huber, *On the Parshin-Beĭlinson adèles for schemes*, Abh. Math. Sem. Univ. Hamburg **61** (1991), 249–273. MR 1138291 (92k:14024)

[Kat86] K. Kato, *Milnor K-theory and the Chow group of zero cycles*, Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983), Contemp. Math., vol. 55, Amer. Math. Soc., Providence, RI, 1986, pp. 241–253. MR 862638 (88c:14012)

[Kat00] Kazuya Kato, *Existence theorem for higher local fields*, Invitation to higher local fields (Münster, 1999), Geom. Topol. Monogr., vol. 3, Geom. Topol. Publ., Coventry, 2000, pp. 165–195. MR 1804933 (2002e:11173)

[Kel99] Bernhard Keller, *On the cyclic homology of exact categories*, J. Pure Appl. Algebra **136** (1999), no. 1, 1–56. MR 1667558 (99m:18012)

[KN] J. Kaad and R. Nest, *Tate tame symbol and the joint torsion of commuting operators*, arXiv:1408.3851.

[KV07] M. Kapranov and E. Vasserot, *Formal loops. II. A local Riemann-Roch theorem for determinantal gerbes*, Ann. Sci. École Norm. Sup. (4) **40** (2007), no. 1, 113–133. MR 2332353 (2008k:14048)

[Lod76] J.-L. Loday, *K-théorie algébrique et représentations de groupes*, Ann. Sci. École Norm. Sup. (4) **9** (1976), 309–377.

[Lura] Jacob Lurie, *Higher algebra*.

[Lurb] Jacob Lurie, *Higher topos theory*.

[Lur09] ———, *Derived algebraic geometry XII: Proper morphisms, completions, and the Grothendieck existence theorem*, arXiv:0911.0018, 11 2009.

[Mer83] A. S. Merkurjev, *On the torsion in K_2 of local fields*, Ann. of Math. (2) **118** (1983), no. 2, 375–381. MR 717828 (85c:11121)

[Mil70] J. Milnor, *Algebraic K-theory and quadratic forms*, Invent. Math. **9** (1969/1970), 318–344. MR 0260844 (41 #5465)

[Mor] Matthew Morrow, *An introduction to higher dimensional local fields and adèles*, arXiv:1204.0586.

[OZ11] D. Osipov and X. Zhu, *A categorical proof of the Parshin reciprocity laws on algebraic surfaces*, Algebra Number Theory **5** (2011), no. 3, 289–337. MR 2833793 (2012h:19010)

[OZ13] ———, *Two-dimensional Contou-Carrère symbol and reciprocity laws*, arXiv:1305.6032, 05 2013.

[Pál10] A. Pál, *On the kernel and the image of the rigid analytic regulator in positive characteristic*, Publ. Res. Inst. Math. Sci. **46** (2010), no. 2, 255–288. MR 2722779 (2011g:19006)

[Par76] A. N. Paršin, *On the arithmetic of two-dimensional schemes. I. Distributions and residues*, Izv. Akad. Nauk SSSR Ser. Mat. **40** (1976), no. 4, 736–773, 949. MR 0419458 (54 #7479)

[Pre11] Luigi Previdi, *Locally compact objects in exact categories*, Internat. J. Math. **22** (2011), no. 12, 1787–1821. MR 2872533

[PSY14] Marco Porta, Liran Shaul, and Amnon Yekutieli, *Completion by derived double centralizer*, Algebr. Represent. Theory **17** (2014), no. 2, 481–494. MR 3181733

[Ros94] J. Rosenberg, *Algebraic K-theory and its applications*, GTM, vol. 147, Springer-Verlag, 1994.

[Saf13] Pavel Safronov, *Quasi-Hamiltonian reduction via classical Chern–Simons theory*, arXiv:1311.6429, 11 2013.

[Sai12] Sho Saito, *On Previdi's delooping conjecture for K-theory*, arXiv:1203.0831, 03 2012.

[Sch03] Peter Schenzel, *Proregular sequences, local cohomology, and completion*, Math. Scand. **92** (2003), no. 2, 161–180. MR 1973941 (2004f:13023)

[Sch04] M. Schlichting, *Delooping the K-theory of exact categories*, Topology **43** (2004), no. 5, 1089–1103. MR 2079996 (2005k:18023)

[Sch06] ———, *Negative K-theory of derived categories*, Math. Z. **253** (2006), no. 1, 97–134. MR 2206639 (2006i:19003)

[Tat68] John Tate, *Residues of differentials on curves*, Ann. Sci. École Norm. Sup. (4) **1** (1968), 149–159. MR 0227171 (37 #2756)

[The] The Stacks Project Authors, *Stacks project*, http://math.columbia.edu/algebraic_geometry/stacks-git.

[TT90] R. W. Thomason and T. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift, Vol. III, Progr. Math., vol. 88, Birkhäuser Boston, Boston, MA, 1990, pp. 247–435. MR 1106918 (92f:19001)

[Wal85] F. Waldhausen, *Algebraic K-theory of spaces*, Algebraic and geometric topology (New Brunswick, N.J., 1983), Lecture Notes in Math., vol. 1126, Springer, Berlin, 1985, pp. 318–419. MR 802796 (86m:18011)

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